

# NORMAL FUNCTIONS AND THE GEOMETRY OF MODULI SPACES OF CURVES

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## 1. INTRODUCTION

A normal function on a complex manifold  $T$  is a special kind of holomorphic section of a bundle  $\mathcal{J}(\mathbb{V}) \rightarrow T$  of compact complex tori constructed from a weight  $-1$  variation of Hodge structure  $\mathbb{V}$  over  $T$ . Normal functions in their modern formulation arose in the work [13] of Griffiths as a tool for understanding algebraic cycles in a complex projective manifold.

In this paper we give two examples to illustrate how normal functions might be a useful, if unconventional, tool for understanding the geometry of moduli spaces of curves. The first is to give a partial answer to a question of Eliashberg, which arose in symplectic field theory [10]. Namely, we compute the class in rational cohomology of the pullback of the 0-section of the universal jacobian  $\mathcal{J}_{g,n}^c \rightarrow \mathcal{M}_{g,n}^c$

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over the moduli space of  $n$ -pointed, stable projective curves of compact type of genus  $g$  along the section defined by

$$F_{\mathbf{d}} : [C; x_1, \dots, x_n] \mapsto \sum_{j=1}^n d_j [x_j] \in \text{Jac } C,$$

where  $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{Z}^n$  satisfies  $\sum_{j=1}^n d_j = 0$ . This section is a normal function.

The second application is to give an alternative and complementary approach to the slope inequalities of the type discovered by Moriwaki [32, 33] such as his result that the divisor class

$$M := (8g + 4)\lambda_1 - g\delta_0 - 4 \sum_{h=1}^{\lfloor g/2 \rfloor} h(g-h)\delta_h$$

on  $\overline{\mathcal{M}}_g$  has non-negative degree on all complete curves in  $\overline{\mathcal{M}}_g$  that do not lie in the boundary divisor  $\Delta := \overline{\mathcal{M}}_g - \mathcal{M}_g$ .<sup>1</sup> (Here the  $\delta_h$  ( $0 \leq h \leq g/2$ ) denote the classes of the components of  $\Delta$ .) This alternative approach leads to actual and conjectural strengthenings of his inequalities: we show that the Moriwaki divisor  $M$  has non-negative degree on all complete curves in  $\mathcal{M}_g^c$  and conjecture that  $M$  has non-negative degree on all complete curves in  $\overline{\mathcal{M}}_g$  that do not lie in the boundary divisor  $\Delta_0 := \overline{\mathcal{M}}_g - \mathcal{M}_g^c$ .

Each normal function section  $\nu$  of  $J(\mathbb{V}) \rightarrow T$  determines a class  $c(\nu)$  in  $H^1(T, \mathbb{V})$ . When  $H^0(T, \mathbb{V}_{\mathbb{Q}})$  vanishes, the normal function  $\nu$  is determined, mod torsion, by its characteristic class  $c(\nu)$ . In such cases, there is a rigid relationship between normal functions and cohomology. When  $T = \mathcal{M}_{g,n}^c$ ,  $g \geq 3$ , and  $\mathbb{V}$  is a variation of Hodge structure corresponding to a non-trivial rational representation of  $\text{Sp}_g$  that does not contain the trivial representation, the group of normal function sections of  $J(\mathbb{V})$  is finitely generated and is known modulo torsion, [15, §8]. This result is recalled in Appendix A. When  $\mathbb{V}$  corresponds to the fundamental representation of  $\text{Sp}_g$ ,  $J(\mathbb{V})$  is the universal jacobian  $\mathcal{J}_{g,n}^c$  over  $\mathcal{M}_{g,n}^c$  and the class of the pullback  $F_{\mathbf{d}}^* \eta_g$  of the zero section of  $\mathcal{J}_{g,n}^c$  can be expressed in terms of the classes of certain basic normal functions defined on  $\mathcal{M}_{g,n}^c$ .

All variations of Hodge structure  $\mathbb{V}$  of geometric origin have a *polarization*; that is, an invariant inner product  $S : \mathbb{V}^{\otimes 2} \rightarrow \mathbb{Q}$  that satisfies the Riemann-Hodge bilinear relations on each fiber. When  $V$  has odd weight, the polarization is skew symmetric. Each invariant, skew-symmetric inner product  $S : \mathbb{V}^{\otimes 2} \rightarrow \mathbb{Q}$  gives rise to a class  $S \circ c(\nu)^2$  in  $H^2(T, \mathbb{Q})$ . It is the image of  $c(\nu)^{\otimes 2}$  under the composition of the cup product with the map induced by  $S$ :

$$H^1(T, \mathbb{V})^{\otimes 2} \xrightarrow{\smile} H^2(T, \mathbb{V}^{\otimes 2}) \xrightarrow{S_*} H^2(T, \mathbb{Q}).$$

The class  $S \circ c(\nu)^2$  has a natural de Rham representative which is a non-negative  $(1, 1)$ -form when  $S$  is a polarization. Moriwaki's inequality for complete curves in  $\mathcal{M}_g^c$  is an immediate consequence of this semi-positivity and the fact that the class of the Moriwaki divisor equals the square  $S \circ c(\nu)^2$  of the class of the most fundamental normal function over  $\mathcal{M}_g$  — viz., the normal function associated to the cycle  $C - C^-$  in  $\text{Jac } C$  that was first studied by Ceresa [7].

<sup>1</sup>A weaker version of this inequality had been proved previously by Cornalba and Harris in [6].

When  $\mathbb{V}$  is a weight  $-1$  polarized variation of Hodge structure over  $T$ , there is a naturally metrized line bundle  $\mathcal{B}$  over  $J(\mathbb{V})$ , which is called the *biextension line bundle*. The curvature of its pullback along a normal function  $\nu : T \rightarrow J(\mathbb{V})$  is the natural de Rham representative of  $S \circ c(\nu)^2$ . This line bundle extends naturally to any compactification  $\overline{T}$  of  $T$ , even if  $\mathbb{V}$  (and hence  $\nu$ ) does not extend to  $\overline{T}$ . This extension is characterized by the property that the metric extends across the codimension 1 strata of  $\overline{T} - T$ . Surprisingly, this extension is *not* natural under pullback to a smooth variety as the metric on the extended line bundle may be singular on strata of codimension  $\geq 2$  of  $\overline{T} - T$ . This curious phenomenon is called *height jumping*. Height jumping and its relevance to refined slope inequalities is discussed in Section 14.

The classes  $c(\nu)$  form part of a larger structure. When  $T = \mathcal{M}_{g,n}$  they should be regarded as twisted tautological cohomology classes. To explain this, we need to introduce a certain graded commutative algebra associated to  $\mathcal{M}_{g,n}$ . Denote the coordinate ring of the symplectic group  $\mathrm{Sp}_g$  by  $\mathcal{O}(\mathrm{Sp}_g)$ . Left and right multiplication induce commuting left and right actions of the mapping class group  $\pi_1(\mathcal{M}_{g,n}, *)$  on it via the standard representation  $\pi_1(\mathcal{M}_{g,n}, x_o) \rightarrow \mathrm{Sp}_g(\mathbb{Q})$ . Using the right action, one obtains a local system  $\mathcal{O}$  of  $\mathbb{Q}$ -algebras over  $\mathcal{M}_{g,n}$ . Since  $\mathcal{O}$  is a local system of commutative  $\mathbb{Q}$ -algebras, its cohomology

$$A_{g,n}^\bullet := H^\bullet(\mathcal{M}_{g,n}, \mathcal{O})$$

is a graded commutative  $\mathbb{Q}$ -algebra. The left  $\mathrm{Sp}_g$ -action on  $\mathcal{O}$  gives it the structure of a graded commutative algebra in the category of  $\mathrm{Sp}_g$ -modules.

The algebraic analogue of the Peter-Weyl Theorem implies that there is an  $\mathrm{Sp}_g \times \mathrm{Sp}_g$ -equivariant isomorphism

$$\mathcal{O}(\mathrm{Sp}_g) \cong \bigoplus_{\lambda} \mathrm{End}(V_{\lambda})^* \cong \bigoplus_{\lambda} V_{\lambda} \boxtimes V_{\lambda}^*,$$

where  $\{V_{\lambda}\}$  is a set of representatives of the isomorphism classes of irreducible  $\mathrm{Sp}_g$ -modules. There is thus an isomorphism

$$A_{g,n}^\bullet \cong \bigoplus_{\lambda} H^\bullet(\mathcal{M}_{g,n}, \mathbb{V}_{\lambda}) \otimes V_{\lambda}^*$$

where  $\mathbb{V}_{\lambda}$  denotes the local system over  $\mathcal{M}_{g,n}$  that corresponds to  $V_{\lambda}$ .

The classes  $c(\nu)$  are more fundamental than their squares  $S \circ c(\nu)^2$ , which are known to be tautological classes. For this reason, we define the tautological subalgebra  $T_{g,n}^\bullet$  of  $A_{g,n}^\bullet$  to be the graded subalgebra generated by the classes  $c(\nu) \otimes V_{\lambda}^*$  of the normal function sections  $\nu$  of the  $J(\mathbb{V}_{\lambda})$ .<sup>2</sup> It is finitely generated. The classification of normal functions [15] over  $\mathcal{M}_{g,n}$ , and the work of Kawazumi and Morita [25] imply that the ring of  $\mathrm{Sp}_g$ -invariants  $(T_{g,n}^\bullet)^{\mathrm{Sp}_g}$  is Faber's tautological ring  $R_{g,n}$  (in cohomology) [11] of  $\mathcal{M}_{g,n}$ . The computations of Morita [31], Kawazumi and Morita [25], and those of this paper, may be regarded as computations in  $T_{g,n}^\bullet$ . For this reason, we propose that the ring  $T_{g,n}^\bullet$  is more fundamental than its  $\mathrm{Sp}_g$ -invariant part  $R_{g,n}$ . It would be interesting to define and study a Chow analogue of  $T_{g,n}^\bullet$ . The significance of this algebra and its relation to normal functions is discussed in Appendix B.

<sup>2</sup>Each  $\mathbb{V}_{\lambda}$  is the local system that underlies a polarized variation of Hodge structure over  $\mathcal{M}_{g,n}$ . It is unique up to Tate twist. The only  $\mathbb{V}_{\lambda}$  that admit non-torsion normal functions have weight  $-1$ .

*Advice to the reader:* Although normal functions have long been a part of algebraic geometry (examples were first considered by Poincaré), they are not currently part of the standard repertoire of modern algebraic geometry. Their modern definition (Definition 5.2), in terms of extensions of variations of Hodge structure, requires an understanding of variations of (mixed) Hodge structure. However, if the reader is prepared to believe that the local systems associated to locally topologically trivial families of algebraic varieties are motivic, and so are variations of mixed Hodge structure, then the definition should be natural.

We assume the reader is familiar with the basic definitions and constructions of Hodge theory. In particular, the reader should know the definition of Hodge structures, mixed Hodge structures, and variations of Hodge structure. The book [35] by Peters and Steenbrink is a good source of basic material on these topics. The paper [18] contains a brief exposition of Schmid's work [40] on the asymptotic properties of variations of Hodge structure that emphasizes the case of degenerations of curves. Finally, the recent survey of normal functions [26] by Kerr and Pearlstein should be a useful supplement, although its emphasis is quite different from that of this article.

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## 2. NOTATION AND CONVENTIONS

All varieties (and stacks) will be defined over the complex numbers. Denote the moduli space of stable  $n$ -pointed curves of genus  $g$  by  $\overline{\mathcal{M}}_{g,n}$ . This is defined when  $2g - 2 + n > 0$  and will be viewed as a stack or as a complex analytic orbifold. As such, it is smooth. Denote the Zariski open subset corresponding to the set of  $n$ -pointed smooth curves by  $\mathcal{M}_{g,n}$  and by  $\mathcal{M}_{g,n}^c$  the Zariski open subset consisting of  $n$ -pointed curves of compact type. Note that  $\mathcal{M}_{g,n}^c = \overline{\mathcal{M}}_{g,n} - \Delta_0$ , where  $\Delta_0$  denotes the boundary divisor of  $\overline{\mathcal{M}}_{g,n}$  whose generic point is an irreducible, geometrically connected curve with one node.

The moduli stack of principally polarized abelian varieties of dimension  $g$  will be denoted by  $\mathcal{A}_g$ . The universal curve of compact type will be denoted by  $\mathcal{C}_g^c \rightarrow \mathcal{M}_g^c$  and its restriction to  $\mathcal{M}_g$  by  $\mathcal{C}_g$ . All of these moduli spaces are globally the quotient of a smooth variety by a finite group, [29, 1].

Vector bundles, variations of (mixed) Hodge structure, etc on a stack  $T$  that is the quotient of a smooth variety  $S$  by a finite group  $G$ , are  $G$ -invariant bundles, variations of (mixed) Hodge structure, etc, over  $S$ . Since all stacks that occur in this paper are of this form, we will not distinguish between stacks and varieties, as working on one of these stacks is working equivariantly on a finite cover that is a variety.

The category of  $\mathbb{Z}$ -mixed Hodge structures by MHS. For  $d \in \mathbb{Z}$ , the Hodge structure of type  $(-d, -d)$  whose underlying lattice is isomorphic to  $\mathbb{Z}$  will be denoted by  $\mathbb{Z}(d)$ . We shall denote by  $\Gamma V$  the set  $\text{Hom}_{\text{MHS}}(\mathbb{Z}(0), V)$  of Hodge classes of type  $(0, 0)$  of the mixed Hodge structure  $V$ . The category of admissible variations of mixed Hodge structure over a smooth variety  $X$  will be denoted by  $\text{MHS}(X)$ .

All cohomology groups will be with  $\mathbb{Q}$  coefficients unless otherwise stated. Similarly, the Chow group of codimension  $d$  cycles on a stack  $X$ , *tensoring with*  $\mathbb{Q}$ , by  $CH^d(X)$ .

### 3. ELIASHBERG'S PROBLEM

To motivate the discussion of normal functions and related topics in subsequent sections, we begin with a brief discussion of the universal jacobian and Eliashberg's problem. Some readers may prefer to begin with Sections 4 and 5. Recall that a stable curve  $C$  of genus  $g$  is of *compact type* if its dual graph is a tree. This condition is equivalent to the condition that its jacobian  $\text{Jac } C := \text{Pic}^0 C$  be an abelian variety.

**3.1. The universal jacobian.** We begin with a review of the transcendental construction of the jacobian of the universal curve over  $\mathcal{M}_g$ . This is a special case of Griffiths' construction of families of intermediate jacobians and normal functions, which are reviewed in Section 5.

First recall the transcendental construction of the jacobian of a smooth projective curve  $C$ , which we recast in the language of Hodge structures. It will be generalized in Section 5 where Griffiths intermediate jacobians are introduced. The Hodge Theorem implies that

$$(1) \quad H^1(C, \mathbb{C}) \cong H^{1,0}(C) \oplus H^{0,1}(C)$$

where  $H^{1,0}(C)$  denotes the space  $H^0(C, \Omega_C^1)$  of holomorphic 1-forms on  $C$  and  $H^{0,1}(C)$  its complex conjugate, the space of anti-holomorphic 1-forms. The first integral cohomology group  $H^1(C, \mathbb{Z})$  endowed with the decomposition (1) is the prototypical Hodge structure of weight 1. Its dual

$$H_1(C, \mathbb{C}) = H^{-1,0}(C) \oplus H^{0,-1}(C)$$

is a Hodge structure of weight  $-1$ , where  $H^{-p,-q}(C)$  is defined to be the dual of  $H^{p,q}(C)$ . The Hodge filtration

$$H_1(C, \mathbb{C}) = F^{-1}H_1(C) \supset F^0H_1(C) \supset F^1H_1(C) = 0.$$

of  $H_1(C)$  is defined by

$$F^p H_1(C) = \bigoplus_{\substack{s \geq p \\ s+t=-1}} H^{s,t}(C).$$

The projection onto  $H^{1,0}(C)$  induces an isomorphism

$$H_1(C, \mathbb{C})/F^0 \cong H^{-1,0}(C) \cong H^0(C, \Omega_C^1)^* := \text{Hom}_{\mathbb{C}}(H^0(C, \Omega_C^1), \mathbb{C}).$$

The composite

$$H_1(C, \mathbb{Z}) \hookrightarrow H_1(C, \mathbb{C}) \rightarrow H_1(C, \mathbb{C})/F^0 \cong H^0(C, \Omega_C^1)^*$$

is the map that takes the homology class of the 1-cycle  $\gamma$  to the functional

$$\int_\gamma := \left\{ \omega \mapsto \int_\gamma \omega \right\} \in H^0(C, \Omega_C^1)^*.$$

It is injective and its image is a lattice. The jacobian of  $C$  is the quotient

$$\text{Jac } C := H^0(C, \Omega_C^1)^*/H_1(C, \mathbb{Z}) \cong H_1(C, \mathbb{C})/(H_1(C, \mathbb{Z}) + F^0 H_1(C)).$$

Every divisor  $D$  of degree 0 on  $C$  can be written as the boundary  $D = \partial\gamma$  of a real 1-chain  $\gamma$ . The Abel-Jacobi mapping

$$\{\text{divisors of degree 0 on } C\}/\text{rational equivalence} \rightarrow \text{Jac } C$$

is defined by taking the divisor class of the boundary of the 1-chain  $\gamma$  to the functional  $\int_\gamma$ . Abel's Theorem implies that it is a group isomorphism. This construction works equally well when  $C$  is a curve of compact type. In this case, the Hodge structure on  $H_1(C)$  is the direct sum of the Hodge structures of its irreducible components. This construction can also be carried out for families of complete curves, where each fiber is either smooth or of compact type. Below we carry this out for the universal curve of compact type. The family of Hodge structures associated to such a family is an example of a *variation of Hodge structure*.

To the universal curve  $\pi : \mathcal{C}_g^c \rightarrow \mathcal{M}_g^c$  of compact type we associate the variation of Hodge structure

$$\mathbb{H} := R^1\pi_*\mathbb{Z}$$

and the corresponding holomorphic vector bundle  $\mathcal{H} := \mathbb{H} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{M}_g^c}$ . The fiber of  $\mathcal{H}$  over the moduli point of  $C$  is  $H^1(C, \mathbb{C})$ . This bundle has a flat holomorphic connection  $\nabla$ . The local monodromy transformations about the divisor  $\Delta_0 := \overline{\mathcal{M}}_{g,n} - \mathcal{M}_{g,n}^c$  are given by the Picard-Lefschetz formula and are therefore unipotent. Consequently,  $\mathcal{H}$  has a canonical extension (in the sense of Deligne [8]) to a vector bundle  $\overline{\mathcal{H}}$  over  $\overline{\mathcal{M}}_g$ .<sup>3</sup> It is characterized by the property that the connection is regular and that its residue at each smooth point of  $\Delta_0$  is nilpotent. Since the monodromy of  $\mathbb{H}$  about  $\Delta_0$  is non-trivial, the local system  $\mathbb{H}$  does not extend across  $\Delta_0$ . Consequently,  $\mathcal{M}_{g,n}^c$  is the maximal Zariski open subset of  $\overline{\mathcal{M}}_{g,n}$  to which  $\mathbb{H}$  extends.

The *Hodge bundle*  $\mathcal{F} := \mathcal{F}^1\mathcal{H}$  is the sub-bundle of  $\mathcal{H}$  whose fiber over the moduli point of  $C$  is  $H^{1,0}(C)$ . It is holomorphic and extends, by a result of Deligne [8], to a holomorphic sub-bundle of  $\overline{\mathcal{F}}^1$  of  $\overline{\mathcal{H}}$ .<sup>4</sup> (See [18, §4] for an exposition.) There is a natural projection  $\check{\mathcal{F}} \rightarrow \mathcal{J}_g$  from the dual of the Hodge bundle to the universal jacobian, which is a covering map on each fiber. The kernel of the projection

$$\check{\mathcal{F}}_C \rightarrow \mathcal{J}_{g,C}$$

<sup>3</sup>See [18] for a concise exposition.

<sup>4</sup>The fiber  $\mathcal{F}_C$  of the Hodge bundle over the stable curve  $C$  can be described as follows. Denote the normalization of  $C$  by  $\nu : \tilde{C} \rightarrow C$ . Let  $D \subset \tilde{C}$  be the inverse image of the double points of  $C$ . Then  $\mathcal{H}_C$  is the subset of  $H^0(\tilde{C}, \Omega_{\tilde{C}}^1(D))$  consisting of those  $w$  such that  $\text{Res}_P w + \text{Res}_Q w = 0$  whenever  $P \neq Q$  and  $\nu(P) = \nu(Q)$ . It is naturally isomorphic to  $F^1 H^1(C_{\vec{v}})$  where  $H^1(C_{\vec{v}})$  denotes the limit MHS on the first order smoothing  $C_{\vec{v}}$  of  $C$  associated to a tangent vector  $\vec{v}$  of  $\overline{\mathcal{M}}_g$  at  $[C]$  which is not tangent to the boundary divisor.

at the moduli point of the stable curve  $C$  is  $H_1(C - C^{\text{sing}}, \mathbb{Z})$ , the first integral homology of the set of smooth points of  $C$ ; the inclusion  $H_1(C - C^{\text{sing}}, \mathbb{Z}) \hookrightarrow \check{\mathcal{F}}_C$  is the integration map. The restriction of  $\mathcal{J}_g$  to  $\overline{\mathcal{M}}_g - \Delta_0^{\text{sing}}$  is a Hausdorff complex analytic orbifold, a fact which follows, for example, from [44, Prop. 2.9]. It is the analytic orbifold associated to the restriction of the universal  $\text{Pic}^0$  stack over  $\overline{\mathcal{M}}_g$  to  $\overline{\mathcal{M}}_g - \Delta_0^{\text{sing}}$ , which is constructed in [3].

Observe that the fiber of  $\mathcal{J}_g \rightarrow \overline{\mathcal{M}}_g$  over the moduli point of a stable curve  $C$  is an abelian variety if and only if  $C$  is of compact type. From the construction, it is clear that the normal bundle of the zero section of  $\mathcal{J}_g$  is the dual  $\check{\mathcal{F}}$  of the Hodge bundle.

**3.2. Eliashberg's question.** Suppose that  $2g - 2 + n > 0$ . Given an integer vector  $\mathbf{d} = (d_1, \dots, d_n)$  with  $\sum_j d_j = 0$ , we have the *rational* section  $F_{\mathbf{d}}$

$$\begin{array}{ccc} & & \mathcal{J}_g \\ & \nearrow F_{\mathbf{d}} & \downarrow \\ \overline{\mathcal{M}}_{g,n} & \longrightarrow & \overline{\mathcal{M}}_g \end{array}$$

of the universal jacobian defined by

$$F_{\mathbf{d}} : [C; x_1, \dots, x_d] \mapsto \left[ \sum_{j=1}^n d_j x_j \right] \in \text{Jac } C$$

when  $C$  is smooth. It is holomorphic over  $\overline{\mathcal{M}}_{g,n} - \Delta_0^{\text{sing}}$ , the complement of the singular locus of  $\Delta_0$ .

Denote the class of the zero section of  $\mathcal{J}_g$  in  $H^{2g}(\mathcal{J}_g, \mathbb{Q})$  by  $\eta_g$ .

*Problem 3.1* (Eliashberg). Compute the class in  $H^{2g}(\overline{\mathcal{M}}_{g,n} - \Delta_0^{\text{sing}})$  of the pullback  $F_{\mathbf{d}}^* \eta_g$  of the zero section of  $\mathcal{J}_g$ .

Denote the  $j$ th Chern class of the Hodge bundle by  $\lambda_j$ . When all  $d_j$  are zero, the section  $F_{\mathbf{d}}$  is defined on all of  $\overline{\mathcal{M}}_{g,n}$ . Since the normal bundle of the zero section is  $\check{\mathcal{F}}$ , the dual of the Hodge bundle, we have:

**Proposition 3.2.** *If  $\mathbf{d} = 0$ , then  $F_{\mathbf{d}}^* \eta_g = (-1)^g \lambda_g \in H^{2g}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ .*

*Remark 3.3.* This result also holds in the Chow ring.

#### 4. FAMILIES OF COMPACT TORI

The restriction of the universal jacobian to  $\mathcal{M}_{g,n}^c$  is a family of compact tori. This section is a discussion of some general properties of families of compact tori.

**Definition 4.1.** A family of compact (real)  $r$ -dimensional tori is a smooth fiber bundle  $f : T \rightarrow B$ , each of whose fibers is a compact, connected abelian Lie group. This bundle is locally (but typically not globally) trivial as a bundle of Lie groups.

We shall assume throughout that  $B$  is connected. The identity section will be denoted by  $s : B \rightarrow T$ . Denote the fiber of  $T$  over  $b \in B$  by  $T_b$ .

For a coefficient ring  $R$ , denote the local system over  $B$  whose fiber over  $b$  is  $H_1(T_b, R)$  by  $\mathbb{H}_R$ . The following assertion is easily proved.

**Proposition 4.2.** *If  $f : T \rightarrow B$  is a family of compact tori, then there is a natural bijection*

$$T \rightarrow \mathbb{H}_{\mathbb{R}}/\mathbb{H}_{\mathbb{Z}}$$

*which commutes with the projections to  $B$  and is a group homomorphism on each fiber.*  $\square$

The flat structure on  $\mathbb{H}_{\mathbb{R}}$  descends to a flat structure on  $T = \mathbb{H}_{\mathbb{R}}/\mathbb{H}_{\mathbb{Z}}$ .

**Corollary 4.3.** *Every bundle of compact tori has a natural flat structure in which the torsion multi-sections are leaves. Equivalently, a bundle of compact real tori has a natural trivialization over each contractible subset of  $B$  in which the torsion sections are constant.*  $\square$

**4.1. The class of a section.** Each section  $s$  of a family  $T \rightarrow B$  of compact tori determines a class  $c(s) \in H^1(B, \mathbb{H}_{\mathbb{Z}})$ . We review three standard constructions of this class.

The first is sheaf theoretic. Denote the sheaf of  $C^\infty$  real-valued functions on  $B$  by  $\mathcal{E}_B$ . The flat vector bundle associated to  $\mathbb{H}_{\mathbb{R}}$  has sheaf of sections  $\mathcal{H} := \mathbb{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{E}_B$ . Denote the sheaf of smooth sections of  $T \rightarrow B$  by  $\mathcal{T}$ . Then one has a short exact sequence

$$0 \rightarrow \mathbb{H}_{\mathbb{Z}} \rightarrow \mathcal{H} \rightarrow \mathcal{T} \rightarrow 0$$

of sheaves. Taking cohomology yields the exact sequence

$$0 \rightarrow H^0(B, \mathbb{H}_{\mathbb{Z}}) \rightarrow H^0(B, \mathcal{H}) \rightarrow H^0(B, \mathcal{T}) \xrightarrow{c} H^1(B, \mathbb{H}_{\mathbb{Z}}) \rightarrow 0.$$

The connecting homomorphism is well defined up to a sign. With the appropriate choice it takes a section  $s$  to its characteristic class. Note that the vanishing of  $c(s)$  implies that  $s$  is homotopic to the zero section. Thus  $H^1(B, \mathbb{H}_{\mathbb{Z}})$  can be identified with the group of homotopy classes of smooth sections of  $T \rightarrow B$ .

The second description is obtained by regarding  $H^1(B, \mathbb{H}_{\mathbb{Z}})$  as congruence classes of extensions

$$0 \rightarrow \mathbb{H}_{\mathbb{Z}} \rightarrow \mathbb{E} \rightarrow \mathbb{Z}_B \rightarrow 0$$

of local systems over  $B$ . Given a section  $s$  of  $T \rightarrow B$ , we can construct such an extension  $\mathbb{E}$  as the local system whose fiber over  $b \in B$  is  $H_1(\widehat{T}_b, \mathbb{Z})$ , where

$$\widehat{T}_b := T_b \cup_h [0, 1]$$

where  $h(0) = 0$  and  $h(1) = s(b)$ . There is a short exact sequence

$$0 \rightarrow H_1(T_b) \rightarrow H_1(\widehat{T}_b) \rightarrow \mathbb{Z} \rightarrow 0$$

in which  $H_1([0, 1], \{0, 1\})$  is identified with  $\mathbb{Z}$  by taking the generator to be the class of a path from 1 to 0.

When  $s(b) \neq 0$ ,  $H_1(\widehat{T}_b) \cong H_1(T_b, \{0, s(b)\}) \cong H^{r-1}(T_b - \{0, s(b)\})$ . The first description of  $c$  is determined only up to a sign. This description fixes the sign.

The third description uses de Rham cohomology. Each  $b \in B$  has an open neighbourhood  $U$  where  $s$  lifts to a section  $\tilde{s} : U \rightarrow \mathcal{H}$  of the flat vector bundle  $\mathcal{H}$ . When  $U$  is connected, such a lift is unique up to translation by a local section of  $\mathbb{H}_{\mathbb{Z}}$ . The 1-form  $d\tilde{s}$  on  $U$  with values in  $\mathbb{H}_{\mathbb{R}}$  is therefore independent of the choice of the lift  $\tilde{s}$ . The de Rham representative of  $c(s)$  is the class that is locally represented by  $d\tilde{s}$ .



4.1.1. *Equivalence of these constructions.* Here is a quick sketch of the equivalence of these definitions. Choose an open covering  $\mathcal{U} = \{U_\alpha\}$  of  $T$  such that  $U_{\alpha_0} \cap \cdots \cap U_{\alpha_q}$  is contractible for all multi-indices  $(\alpha_0, \dots, \alpha_q)$ . Such an open covering can be constructed by taking each  $U_\alpha$  to be a geodesically convex ball with respect to some riemannian metric on  $T$ . The complex  $C^\bullet(\mathcal{U}, \mathcal{F})$  of Čech cochains with coefficients in  $\mathcal{F}$  computes  $H^\bullet(T, \mathcal{F})$  when  $\mathcal{F}$  is  $\mathbb{H}_\mathbb{Z}$ ,  $\mathbb{H}_\mathbb{R}$ ,  $\mathcal{H}$  and  $\mathcal{T}$ . Suppose that  $s$  is a smooth section of  $T \rightarrow B$ . Its restriction to  $U_\alpha$  can be lifted to a section  $s_\alpha$  of  $\mathcal{H}$ . The difference  $c_{\alpha\beta} := s_\beta - s_\alpha$  is a section of  $\mathbb{H}_\mathbb{Z}$  over  $U_\alpha \cap U_\beta$ . The class of  $c(s)$  is represented by the cocycle  $c_{\alpha\beta} \in C^1(\mathcal{U}, \mathbb{H}_\mathbb{Z})$ .

The class of an extension

$$0 \rightarrow \mathbb{H}_\mathbb{Z} \rightarrow \mathbb{E} \rightarrow \mathbb{Z}_T \rightarrow 0.$$

is computed by choosing sections  $e_\alpha$  of  $\mathbb{E} \rightarrow \mathbb{Z}_T$  over each  $U_\alpha$ . The class of the extension is represented by the cocycle  $(e_\beta - e_\alpha)_{\alpha\beta} \in C^1(\mathcal{U}, \mathbb{H}_\mathbb{Z})$ . In the case above, one can take the local section  $e_\alpha$  to be the class in  $H_1(T_b, \{0, s(b)\})$  of the image of the path in the universal covering  $\mathbb{H}_{\mathbb{R}, b}$  of  $T_b$  that goes from 0 to  $s_\alpha(b)$ . Then  $e_\beta - e_\alpha = c_{\alpha\beta}$ , as required.

Denote the de Rham sheaf of smooth  $\mathbb{R}$ -valued forms on  $T$  by  $\mathcal{E}_T^\bullet$ . Standard arguments imply that the inclusions

$$C^\bullet(\mathcal{U}, \mathbb{H}_\mathbb{R}) \hookrightarrow C^\bullet(\mathcal{U}, \mathcal{E}_T^\bullet \otimes \mathbb{H}_\mathbb{R}) \hookleftarrow \mathcal{E}_T^\bullet \otimes \mathbb{H}_\mathbb{R}$$

induce isomorphisms on homology. The standard zig-zag argument implies that the class of the cocycle  $(c_{\alpha\beta})$  is represented by the element of

$$E^\bullet(T, \mathbb{H}_\mathbb{R}) := H^0(T, \mathcal{E}_T^\bullet \otimes \mathbb{H}_\mathbb{R})$$

whose restriction to  $U_\alpha$  is  $ds_\alpha$ .

**4.2. Invariant cohomology classes.** The flat structure of a family of compact tori  $T \rightarrow B$  can be used to construct natural de Rham representatives of cohomology classes on  $T$ . We will say that a differential form  $w$  on a manifold  $M$  with a foliation  $\mathcal{L}$  is *parallel with respect to  $\mathcal{L}$*  if the Lie derivative of  $w$  with respect to each vector field tangent to  $\mathcal{L}$  vanishes. A family of tori is foliated as it is a flat family of tori.

The following lemma is proved in [23, Lemma 5.1].

**Lemma 4.4.** *If  $f : T \rightarrow B$  is a family of compact tori, there is a natural mapping*

$$\sigma : H^0(B, R^k f_* \mathbb{R}) \rightarrow H^k(T, \mathbb{R})$$

*whose composition with the projection*

$$H^k(T, \mathbb{R}) \rightarrow H^0(B, R^k f_* \mathbb{R})$$

*is the identity. Moreover, for each  $u \in H^0(B, R^k f_* \mathbb{R})$ , the extended class  $\sigma(u)$  has a natural differential form representative  $w_u$  whose restriction to each fiber is a closed, translation-invariant differential form, and which is parallel with respect to the flat structure. This class has the property that its restriction to every leaf (such as the zero section and every torsion multi-section) is zero.*  $\square$

**4.3. The Poincaré dual of the zero section.** Let  $r$  be the real dimension of the fiber of  $f : T \rightarrow B$ . If  $B$  and  $T$  are oriented and  $B$  is connected, then

$$H^0(B, R^r f_* \mathbb{R}) \cong \mathbb{R}.$$

Let  $u$  be the element of this group whose value on one (and hence all) fibers is 1. Denote the class  $\sigma(u) \in H^r(T, \mathbb{R})$  by  $\psi$ .

**Proposition 4.5.** *If the base  $B$  is a compact manifold (possibly with boundary), then the Poincaré dual of the zero section is  $\psi$ .*

*Proof.* Set  $d = \dim_{\mathbb{R}} B$ . For  $e \in \mathbb{Z}$  define  $[e] : T \rightarrow T$  to be the map whose restriction to each fiber is multiplication by  $e$ . Since  $[e]$  induces multiplication by  $e^k$  on  $R^k f_* \mathbb{Q}$ , it follows that the Leray spectral sequence degenerates at  $E_2$ . It also follows that the eigenvalues of the induced mapping  $[e]^*$  on  $H^k(T)$  lie in  $\{1, e, \dots, e^k\}$ . Since none of these eigenvalues is zero when  $e \neq 0$ ,  $[e]^*$  is invertible on rational cohomology when  $e \neq 0$ . Similar assertions hold for the homology spectral sequence

$$H_s(B, \partial B; \mathbb{H}_t) \implies H_{s+t}(T, \partial T),$$

where  $\mathbb{H}_t$  denotes the local system whose fiber over  $b \in B$  is  $H_t(T_b)$ . Since

$$H_{r+d}(T, \partial T) = H_d(B, \partial B; \mathbb{H}_r)$$

it follows that  $[e]_*[T] = e^r[T]$ , where  $[T]$  denotes the fundamental class of  $(T, \partial T)$ .

Now assume that  $e > 1$ . Then the collapsing of the Leray spectral sequence implies that the dimension of the  $e^r$ -eigenspace of  $H^r(T)$  is one. Since the form  $w_\psi$  that naturally represents  $\psi$  has the property that

$$[e]^* w_\psi = e^r w_\psi,$$

and since  $\psi$  is non-trivial (as it has non trivial integral over a fiber), it follows that  $\psi$  spans this eigenspace.

Note that the class  $[Z] \in H_d(T, \partial T)$  of the zero-section is an eigenvector of  $[e]_*$  with eigenvalue 1. Denote the Poincaré dual of the zero section by  $\eta_Z$ . It is characterized by the property that

$$[T] \cap \eta_Z = [Z] \in H_d(T, \partial T),$$

where  $\cap$  denotes the cap product [41, p. 254]

$$\cap : H_{d+r}(T, \partial T) \otimes H^r(T) \rightarrow H_d(T, \partial T).$$

Since  $e_*[Z] = [Z]$ , standard properties of the cap product [41, Assertion 16, p. 254], we have

$$[e]_*([T] \cap [e]^* \eta_Z) = ([e]_*[T]) \cap \eta_Z = e^r[T] \cap \eta_Z = [e]_*([T] \cap e^r \eta_Z).$$

Since  $[e]_*$  and capping with  $[T]$  are both isomorphisms, it follows that  $[e]^* \eta_Z = e^r \eta_Z$ . Since  $\eta_Z$  and  $\psi$  both lie in the  $e^r$ -eigenspace and agree on each fiber, they are equal.  $\square$

Since the normal bundle of the zero section  $Z$  is the flat bundle  $\mathbb{H}_{\mathbb{R}}$ , the Euler class of the normal bundle of  $Z$  vanishes in rational cohomology.

**Corollary 4.6.** *The restriction of the Poincaré dual of the zero section  $Z$  of a family of compact tori to  $Z$  vanishes in rational cohomology.*  $\square$

Combined with Proposition 3.2, this implies the well-known property of the top Chern class of the Hodge bundle.

**Corollary 4.7.** *The restriction of  $\lambda_g$  to  $\mathcal{M}_g^c$  vanishes in rational cohomology.*

4.4. **The class  $S \circ c(s)^2$ .** Suppose that  $f : T \rightarrow B$  is a flat family of tori and that  $\mathbb{H}$  is the corresponding local system such that  $T = \mathbb{H}_{\mathbb{R}}/\mathbb{H}_{\mathbb{Z}}$ . A flat, skew-symmetric inner product  $S : \mathbb{H}_{\mathbb{R}} \otimes \mathbb{H}_{\mathbb{R}} \rightarrow \mathbb{R}$  gives an element of  $H^0(T, R^2 f_* \mathbb{R}_T)$ . Lemma 4.4 implies that  $S$  determines a closed 2-form  $\phi_S$  on  $T$ . It is characterized by the properties:

- (i) its restriction to the fiber  $T_b$  of  $T$  is the translation invariant 2-form on  $T_b$  that corresponds to  $S$ ,
- (ii) it is parallel with respect to the flat structure on  $T$ ,
- (iii) its restriction to the zero-section of  $T$  is zero.

The following result is easily proved using the de Rham description of  $c(s)$ .

**Proposition 4.8.** *If  $s$  is a holomorphic section of  $T \rightarrow B$ , then the form  $s^* \phi_S$  represents the cohomology class  $S \circ c(s)^2 \in H^2(T, \mathbb{R})$ .  $\square$*

## 5. NORMAL FUNCTIONS

Normal functions are our primary tool. They are holomorphic sections of families of intermediate jacobians that satisfy certain infinitesimal and asymptotic conditions. In this section, we recall Griffiths construction of intermediate jacobians and of the normal function associated to a family of homologically trivial algebraic cycles in a family of smooth projective varieties.

5.1. **Intermediate Jacobians.** Suppose that  $Y$  is a compact Kähler manifold and that  $Z$  is an algebraic  $d$ -cycle in  $Y$  where  $0 \leq d < \dim Y$ . One has the exact sequence

$$0 \rightarrow H_{2d+1}(Y) \rightarrow H_{2d+1}(Y, \mathbb{Z}) \rightarrow H_{2d}(|Z|) \rightarrow H_{2d}(Y) \rightarrow \cdots$$

of integral homology groups associated to the pair  $(Y, |Z|)$ , where  $|Z|$  denotes the support of  $Z$ . It is an exact sequence of mixed Hodge structure. The class of the cycle  $Z$  defines a morphism of mixed Hodge structures

$$c_Z : \mathbb{Z}(d) \rightarrow H_{2d}(|Z|),$$

where  $\mathbb{Z}(d)$  denotes the Hodge structure of type  $(-d, -d)$  whose underlying lattice is isomorphic to  $\mathbb{Z}$ . If  $Z$  is null homologous, we can pull back the above sequence along  $c_Z$  to obtain an extension

$$0 \rightarrow H_{2d+1}(Y) \rightarrow E_Z \rightarrow \mathbb{Z}(d) \rightarrow 0$$

in MHS, the category of mixed Hodge structures. Tensoring with  $\mathbb{Z}(-d)$  gives an extension

$$0 \rightarrow H_{2d+1}(Y, \mathbb{Z}(-d)) \rightarrow E_Z(-d) \rightarrow \mathbb{Z}(0) \rightarrow 0$$

and thus a class  $e_Z$  in

$$\text{Ext}_{\text{MHS}}^1(\mathbb{Z}(0), H_{2d+1}(Y, \mathbb{Z}(-d))).$$

Note that, since  $H_{2d+1}(Y)$  has weight  $-(2d+1)$ ,  $H_{2d+1}(Y, \mathbb{Z}(-d))$  has weight  $-1$ .

Suppose that  $V$  is a Hodge structure of weight  $-1$  whose underlying lattice  $V_{\mathbb{Z}}$  is torsion free. The associated jacobian

$$J(V) := V_{\mathbb{C}} / (V_{\mathbb{Z}} + F^0 V_{\mathbb{C}})$$

is a compact complex torus. In general,  $J(V)$  is not an abelian variety. When  $V$  is the weight  $-1$  Hodge structure  $H_{2d+1}(Y, \mathbb{Z}(-d)) \pmod{\text{its torsion}}$ ,  $J(V)$  is the  $d$ th Griffiths intermediate jacobian of  $Y$ .

There is a natural isomorphism (see [4] or [35, §3.5], for example)

$$\mathrm{Ext}_{\mathrm{MHS}}^1(\mathbb{Z}(0), V) \cong J(V).$$

The class  $e_Z$  of a homologically trivial  $d$ -cycle  $Z$  in  $Y$  can thus be viewed as a class

$$e_Z \in J(H_{2d+1}(Y, \mathbb{Z}(d)))$$

in the  $d$ th Griffiths intermediate jacobian. This class can be described explicitly by Griffiths' generalization [13] of the Abel-Jacobi construction, which we now recall. First observe that the standard pairing between  $H_{2d+1}(Y)$  and  $H^{2d+1}(Y)$  induces an isomorphism

$$H_{2d+1}(Y, \mathbb{Z}(-d))/F^0 \cong \mathrm{Hom}_{\mathbb{C}}(F^{d+1}H^{2d+1}(Y), \mathbb{C}).$$

The natural mapping  $H_{2d+1}(Y, \mathbb{Z}(-d)) \rightarrow H_{2d+1}(Y, \mathbb{Z}(-d))/F^0$  corresponds to the integration mapping

$$H_{2d+1}(Y, \mathbb{Z}) \rightarrow \mathrm{Hom}_{\mathbb{C}}(F^{d+1}H^{2d+1}(Y), \mathbb{C})$$

that takes the homology class  $z$  to  $\xi \mapsto \int_z \xi$ . A homologically trivial  $d$ -cycle  $Z$  in  $Y$  can be written as the boundary  $\partial\Gamma$  of a (topological)  $(2d+1)$ -chain  $\Gamma$ . Note that  $\Gamma$  is well defined up to the addition of an integral  $(2d+1)$ -cycle. Classical Hodge theory implies that each element  $u$  of  $F^{d+1}H^{2d+1}(Y)$  can be represented by a closed  $C^\infty$  form  $\xi_u$  in the  $(d+1)$ st level of the Hodge filtration on the de Rham complex of  $Y$  and that any two such forms differ by the exterior derivative of a form in the same level  $F^{d+1}$  of the de Rham complex. The point  $e_Z$  is represented by the element

$$\int_\Gamma : u \mapsto \int_\Gamma \xi_u.$$

of

$$\mathrm{Hom}_{\mathbb{C}}(F^{d+1}H^{2d+1}(Y), \mathbb{C})/H_{2d+1}(Y, \mathbb{Z}) \cong J(H_{2d+1}(Y, \mathbb{Z}(-d))).$$

Stokes' Theorem implies that the image of this functional in the intermediate jacobian depends only on  $Z$  and not on the choice of  $\Gamma$  or  $\xi_u$ .

This construction generalizes the classical construction for 0-cycles on curves that was sketched in Section 3. More generally, it generalizes the classical construction for 0-cycles, where  $J(H_1(Y)) \cong \mathrm{Alb} Y$ , and for divisors, where  $J(H_{2d-1}(Y)) \cong \mathrm{Pic}^0 Y$  and  $d = \dim Y$ .

**5.2. Normal Functions.** Suppose that  $\overline{X}$  is a complex projective manifold and that  $X = \overline{X} - D$  where  $D$  is a normal crossings divisor in  $\overline{X}$ . Suppose that  $\mathbb{V}$  is a variation of Hodge structure over  $X$  of weight  $-1$ . Denote by  $J(\mathbb{V}) \rightarrow X$  the corresponding bundle of intermediate jacobians; the fiber over  $x \in X$  is  $J(V_x)$ , where  $V_x$  is the fiber of  $\mathbb{V}$  over  $x$ . It is a family of compact tori.

The discussion of Section 4.1 implies that a holomorphic section  $\nu : X \rightarrow J(\mathbb{V})$  determines a cohomology class  $c(\nu) \in H^1(X, \mathbb{V})$  and a local system  $\mathbb{E} \rightarrow X$  which is an extension

$$0 \rightarrow \mathbb{V} \rightarrow \mathbb{E} \rightarrow \mathbb{Z}_X \rightarrow 0.$$

The point  $\nu(x) \in J(V_x) \cong \mathrm{Ext}_{\mathrm{MHS}}^1(\mathbb{Z}(0), V_x)$  determines a mixed Hodge structure on the fiber  $E_x$  of  $\mathbb{E}$  over  $x \in X$  so that

$$0 \rightarrow V_x \rightarrow E_x \rightarrow \mathbb{Z}(0) \rightarrow 0$$

is an extension in MHS. That  $\nu$  is holomorphic implies that this family of MHSs varies holomorphically with  $x \in X$ .

**Example 5.1.** Families of homologically trivial algebraic cycles give rise to such extensions. Suppose that  $\overline{Y}$  is a complex projective manifold and that  $f : \overline{Y} \rightarrow \overline{X}$  is a morphism whose restriction to  $X$  is a family  $Y \rightarrow X$  of projective manifolds. Suppose that  $Z$  is an algebraic  $d$ -cycle in  $Y$  such that the restriction  $Z_x$  of  $Z$  to the fiber over each  $x \in X$  is a homologically trivial  $d$ -cycle in  $Y_x$ . Applying the construction of the previous section fiber-by-fiber, one obtains an extension  $\mathbb{E}_Z$  of  $\mathbb{Z}_X(0)$  by the variation of Hodge structure  $\mathbb{V}$  of weight  $-1$  whose fiber over  $x \in X$  is  $V_x = H_{2d+1}(Y_x, \mathbb{Z}(-d))$ . The family  $\{E_x\}_{x \in X}$  of extensions of MHS corresponds to a holomorphic section  $\nu$  of the bundle  $J(\mathbb{V}) \rightarrow X$  of intermediate jacobians.

The section  $\nu$  is not an arbitrary holomorphic section. It satisfies the *Griffiths' infinitesimal period relation*<sup>5</sup> at each  $x \in X$  and also satisfies strong (and technical) conditions as  $Y_x$  degenerates. A succinct way to state these conditions is to say that the corresponding family  $\mathbb{E}$  of MHS is an *admissible variation of MHS* in the sense Steenbrink-Zucker [42] and Kashiwara [24]. (The definition and an exposition of admissible variations of MHS can be found in [35, §14.4.1].)

All “naturally defined local systems” over a smooth variety  $X$  that arise from families of varieties over  $X$ , such as the one constructed in Example 5.1, are admissible variations of MHS.<sup>6</sup> The admissibility conditions axiomatize the infinitesimal and asymptotic properties satisfied by such geometrically defined local systems.

**Definition 5.2** (Hain [15, §6], Saito [39]). A section  $\nu$  of a family  $J(\mathbb{V}) \rightarrow X$  of intermediate jacobians is a *normal function* when the corresponding family of MHS  $\mathbb{E}$  is an admissible variation of MHS.

The preceding discussion implies that the section  $\nu$  of  $J(\mathbb{V}) \rightarrow X$  associated to a family of null homologous cycles is a normal function. Several concrete examples of normal functions over moduli spaces of curves will be given in Section 5.5. A detailed discussion of normal functions can be found in [26, §2.11], where they are called *admissible* normal functions.

Denote the category of admissible variations of mixed Hodge structure over a smooth variety  $X$  by  $\text{MHS}(X)$ . It is abelian. The definition of normal functions implies that normal function sections of  $J(\mathbb{V}) \rightarrow X$  correspond to elements of  $\text{Ext}_{\text{MHS}(X)}^1(\mathbb{Z}_X(0), \mathbb{V})$ . In the appendix to [19] it is proved that one has an exact sequence

$$(2) \quad 0 \rightarrow \text{Ext}_{\text{MHS}}^1(\mathbb{Z}(0), H^0(X, \mathbb{V}_{\mathbb{Z}})) \xrightarrow{j} \text{Ext}_{\text{MHS}(X)}^1(\mathbb{Z}_X(0), \mathbb{V}_{\mathbb{Z}}) \xrightarrow{\delta} H^1(X, \mathbb{V}_{\mathbb{Z}})$$

where  $\delta$  takes a normal function  $\nu$  to its class  $c(\nu)$ . An immediate consequence is the following rigidity property of normal functions.

**Proposition 5.3.** *If  $H^0(X, \mathbb{V}_{\mathbb{Q}}) = 0$ , then the group  $\text{Ext}_{\text{MHS}(X)}^1(\mathbb{Z}(0)_X, \mathbb{V}_{\mathbb{Z}})$  is finitely generated and each normal function section  $\nu$  of  $J(\mathbb{V})$  is determined, up to a torsion section, by its class  $c(\nu) \in H^1(X, \mathbb{V})$ .*

**5.3. Extending normal functions.** The following result guarantees that the normal functions that we will define over  $\mathcal{M}_{g,n}$  extend to  $\mathcal{M}_{g,n}^c$ . A proof can be found, for example, in [15, Thm. 7.1].

<sup>5</sup>This states that if  $\tilde{\nu}$  is a local holomorphic lift of a normal function  $\nu$  to a section of  $\mathbb{E} \otimes \mathcal{O}_X$ , then its derivative  $\nabla \tilde{\nu}$  is a section of  $\mathcal{F}^{-1}(\mathbb{E} \otimes \mathcal{O}_X) \otimes \Omega_X^1$ .

<sup>6</sup>These results are due to Steenbrink-Zucker [42], Navarro et al [12] and Saito [38]. Precise statements can be found in [35, Thm. 14.51].

**Proposition 5.4.** *Suppose that  $\mathbb{V}$  is a variation of Hodge structure of weight  $-1$  over a smooth variety  $X$ . If  $\nu$  is a normal function section of  $J(\mathbb{V})$  that is defined on the complement  $X - Y$  of a closed subvariety  $Y$  of  $X$ , where  $Y \neq X$ , then  $\nu$  extends across  $Y$  to a normal function section of  $J(\mathbb{V})$  over  $X$ .*

Note that, in this result, the variation  $\mathbb{V}$  is defined over  $X$ , not just over  $X - Y$ . The proposition asserts that normal functions defined generically on  $X$  extend to  $X$ . It does not assert that if  $\mathbb{V}$  is defined only over  $X - Y$ , then a normal function section of  $J(\mathbb{V})$  over  $X - Y$  will extend to  $X$ . Before discussing this problem, one first has to construct an extension of  $J(\mathbb{V})$  to  $X$ . The existence of such extensions is discussed in [44] and [26], for example.

**5.4. Some variations of Hodge structure.** Denote the moduli stack of principally polarized abelian varieties of dimension  $g$  by  $\mathcal{A}_g$ . Here we introduce three variations of Hodge structure over  $\mathcal{A}_g$  whose pullbacks to  $\mathcal{M}_{g,n}^c$  have a special geometric significance. Throughout we suppose that  $g \geq 2$ .

Denote the universal abelian variety over  $\mathcal{A}_g$  by  $f : \mathcal{X} \rightarrow \mathcal{A}_g$ . The local system

$$\mathbb{H} := R^1 f_* \mathbb{Z}_{\mathcal{X}}(1)$$

is a variation of Hodge structure over  $\mathcal{A}_g$  of weight  $-1$ . Its pullback to  $\mathcal{M}_{g,n}^c$  along the period mapping  $\mathcal{M}_{g,n}^c \rightarrow \mathcal{A}_g$  will also be denoted by  $\mathbb{H}$ . The corresponding family of intermediate jacobians  $J(\mathbb{H})$  over  $\mathcal{A}_g$  is naturally isomorphic to  $\mathcal{X}$ , the universal principally polarized abelian variety.

The construction of the universal jacobian  $\mathcal{J}_g$  in Section 3 implies that its restriction to  $\mathcal{M}_g^c$  is a bundle of intermediate jacobians.

**Proposition 5.5.** *If  $g \geq 2$ , then  $J(\mathbb{H}) \rightarrow \mathcal{A}_g$  is naturally isomorphic to the universal abelian variety and its pullback to  $\mathcal{M}_g^c$  is isomorphic to the universal jacobian  $\mathcal{J}_g^c$ .*

The canonical polarization defines a morphism  $S_H : \mathbb{H} \otimes \mathbb{H} \rightarrow \mathbb{Z}(1)$  into the constant variation of Hodge structure  $\mathbb{Z}(1)$ . It can be regarded as a section of  $(\Lambda^2 \mathbb{H})(-1)$ , which we shall also denote by  $S_H$ .

Denote by  $\mathbb{L}$  the variation of Hodge structure  $(\Lambda^3 \mathbb{H})(-1)$ . It has weight  $-1$ . Wedging with  $S_H$  defines an inclusion

$$\mathbb{H} \hookrightarrow \mathbb{L}, \quad x \mapsto x \wedge S_H$$

of variations of Hodge structure. Set  $\mathbb{V} = \mathbb{L}/\mathbb{H}$ . Note that when  $g = 2$ ,  $\mathbb{L} \cong \mathbb{H}$  and  $\mathbb{V} = 0$ . Denote the fibers of  $\mathbb{H}$ ,  $\mathbb{L}$  and  $\mathbb{V}$  over the moduli point of  $\text{Jac } C$  by  $H_C$ ,  $L_C$  and  $V_C$ , respectively.

*Remark 5.6.* The variation  $\mathbb{L}$  over  $\mathcal{M}_g^c$  is isomorphic to the variation  $R^3 \pi_* \mathbb{Z}(2)$  where  $\pi : \mathcal{J}_g^c \rightarrow \mathcal{M}_g^c$  denotes the projection. Its fiber over  $[C]$  is  $H^3(\text{Jac } C, \mathbb{Z}(2))$ . The twist  $\mathbb{Z}(2)$  lowers the weight from 3 to  $-1$ . Its quotient  $\mathbb{V}$  is an integral form of the  $\mathbb{Q}$ -variation of Hodge structure whose fiber over  $[C]$  is the primitive part of  $H^3(\text{Jac } C, \mathbb{Q}(2))$ .

**5.5. Normal Functions over  $\mathcal{M}_g^c$  and  $\mathcal{M}_{g,n}^c$ .** Suppose that  $g \geq 1$ . An irreducible representation  $V$  of  $\text{Sp}_g$  determines a variation of Hodge structure  $\mathbb{V}$  over  $\mathcal{M}_{g,n}^c$  that is unique up to Tate twist.<sup>7</sup> Since every such variation of Hodge structure  $\mathbb{V}$  extends

<sup>7</sup>This is very well-known. A proof can be found, for example, in [15, §9].

to  $\mathcal{M}_{g,n}^c$ , Proposition 5.4 implies that every normal function section of  $J(\mathbb{V})$  defined over  $\mathcal{M}_{g,n}$  extends to a normal function on  $\mathcal{M}_{g,n}^c$ .

By Proposition 5.5, the bundle of intermediate jacobians that corresponds to the fundamental representation of  $\mathrm{Sp}_g$  is the pullback of the universal jacobian  $J(\mathbb{H}) = \mathcal{J}_g$  to  $\mathcal{M}_{g,n}^c$ . Its group of sections  $\mathrm{Ext}_{\mathrm{MHS}(\mathcal{M}_{g,1})}^1(\mathbb{Z}(0), \mathbb{H})$  is free of rank 1. Its generator  $\mathcal{K}$  is the normal function that takes the moduli point  $[C, x]$  of the pointed curve  $(C, x)$  to

$$\mathcal{K}([C, x]) := (2g - 2)[x] - K_C \in \mathrm{Jac}(C)$$

where  $K_C$  denotes the canonical divisor class of  $C$ .

When  $n \geq 1$ , we can pull back  $\mathcal{K}$  along the  $j$ th projection  $\mathcal{M}_{g,n} \rightarrow \mathcal{M}_{g,1}$  to obtain the normal function

$$\mathcal{K}_j : \mathcal{M}_{g,n} \rightarrow J(\mathbb{H}) \quad j = 1, \dots, n.$$

Explicitly

$$\mathcal{K}_j([C, x_1, \dots, x_n]) = (2g - 2)[x_j] - K_C \in \mathrm{Jac}(C).$$

When  $n \geq 2$  we also have the normal functions

$$\mathcal{D}_{j,k} : \mathcal{M}_{g,n} \rightarrow J(\mathbb{H}) \quad 1 \leq j < k \leq n$$

defined by

$$\mathcal{D}_{j,k}([C, x_1, \dots, x_n]) = [x_j] - [x_k] \in \mathrm{Jac}(C).$$

**Proposition 5.7** (cf. [15, Thm. 12.3]). *If  $g \geq 3$  and  $n \geq 2$ , then the group of normal function sections (indeed, all sections) of  $J(\mathbb{H}) \rightarrow \mathcal{M}_{g,n}$  is torsion free and is generated by  $\mathcal{K}_1, \dots, \mathcal{K}_n$  and the  $\mathcal{D}_{j,k}$  where  $1 \leq j < k \leq n$ .*

The most interesting normal function over  $\mathcal{M}_{g,n}$  is constructed from the Ceresa cycle in the universal jacobian. Suppose that  $C$  is a smooth projective curve of genus  $g \geq 3$  and that  $x \in C$ . Then one has the imbedding

$$\mu_x : C \rightarrow \mathrm{Jac} C$$

that takes  $y$  to  $[y] - [x]$ . Its image is an algebraic 1-cycle in  $\mathrm{Jac} C$  that we denote by  $C_x$ . Let  $i$  be the involution  $u \mapsto -u$  of  $\mathrm{Jac} C$ . Set  $C_x^- = i_* C_x$ . Since  $i^* : H^1(\mathrm{Jac} C) \rightarrow H^1(\mathrm{Jac} C)$  is multiplication by  $-1$ , it follows that  $i^* : H^k(\mathrm{Jac} C) \rightarrow H^k(\mathrm{Jac} C)$  is multiplication by  $(-1)^k$ . This implies that the 1-cycle  $C_x - C_x^-$ , called the *Ceresa cycle*, is homologically trivial. It therefore determines a class

$$\nu_x(C) \in J(H_3(\mathrm{Jac} C, \mathbb{Z}(-1)))$$

and a normal function:

$$J(\mathbb{L}) \xrightarrow{\tilde{\nu}} \mathcal{M}_{g,1}$$

whose value at  $[C, x]$  is  $\nu_x(C)$ . The inclusion  $\mathbb{H} \hookrightarrow \mathbb{L}$  induces an inclusion

$$j : \mathcal{J}_{g,1} = J(\mathbb{H}) \hookrightarrow J(\mathbb{L}).$$

It is proved in [36] that if  $x, y \in C$ , then

$$\tilde{\nu}_x(C) - \tilde{\nu}_y(C) = 2j([x] - [y]) \in J(L_C)$$

so that the image  $\nu(C)$  of  $\nu_x(C)$  in  $J(V_C)$  does not depend on the choice of  $x \in C$ . This implies that  $\tilde{\nu}$  is pulled back from a normal function

$$J(\mathbb{V}) \xrightarrow{\tilde{\nu}} \mathcal{M}_g.$$

We will abuse notation and also denote its pullback to  $\mathcal{M}_{g,n}$  by  $\nu$ .

**Proposition 5.8** ([15, Thm. 8.3]). *If  $g \geq 3$  and  $n \geq 0$ , then the group of normal function sections of  $J(\mathbb{V}) \rightarrow \mathcal{M}_{g,n}$  is freely generated by  $\nu$ .*

Proposition 5.4 implies that the normal functions  $\nu, \mathcal{K}_j, \delta_{j,k}$  extend canonically to normal functions over  $\mathcal{M}_{g,1}^c$ .

## 6. BIEXTENSION LINE BUNDLES

This section is a brief review of facts about biextension line bundles from [14], [28], and [23]. It is needed to prove that the square of the class of a normal function extends naturally to a class on  $\overline{\mathcal{M}}_{g,n}$  even though the normal function itself does not extend.

Suppose that  $\mathbb{U}$  is a variation of Hodge structure over an algebraic manifold  $X$  of weight  $-1$  endowed with a flat inner product  $S$  that satisfies the condition

$$S \in \text{Hom}_{\text{MHS}}(U_x^{\otimes 2}, \mathbb{Z}(1)) \text{ for all } x \in X.$$

Equivalently,

$$S(U_x^{p,q}, \overline{U_x^{r,s}}) = 0 \text{ unless } p = r \text{ and } q = s.$$

Set  $\check{\mathbb{U}} := \text{Hom}_{\mathbb{Z}}(\mathbb{U}, \mathbb{Z}_X(1))$ . This also a variation of Hodge structure of weight  $-1$ . There is a natural isomorphism

$$\text{Ext}_{\text{MHS}(X)}^1(\mathbb{Z}_X(0), \check{\mathbb{U}}) \cong \text{Ext}_{\text{MHS}(X)}^1(\mathbb{U}, \mathbb{Z}_X(1)).$$

The *biextension line bundle*  $\mathcal{B}$  is a line bundle over  $J(\mathbb{U}) \times_X J(\check{\mathbb{U}})$ . Denote the associated  $\mathbb{G}_m$ -bundle by  $\mathcal{B}^*$ . The fiber of the projection

$$(3) \quad \mathcal{B}^* \rightarrow J(\mathbb{U}) \times_X J(\check{\mathbb{U}}) \rightarrow X$$

over  $x \in X$  is the set of all mixed Hodge structures whose weight graded quotients are  $\mathbb{Z}(0)$ ,  $V_x$  and  $\mathbb{Z}(1)$  via a fixed isomorphism. These are called *biextensions*. A detailed exposition of the construction of  $\mathcal{B}$  is given in [23, §7].

A (Hodge) biextension is a section  $\beta$  of (3) that corresponds to an admissible variation of MHS  $\mathbb{B}$  over  $X$  with weight graded quotients  $\mathbb{Z}_X(0)$ ,  $\mathbb{U}$  and  $\mathbb{Z}_X(1)$ . Its fiber over  $x \in X$  is the biextension  $\beta(x)$ . The composite of a biextension  $\beta$  with the projection  $\mathcal{B}^* \rightarrow J(\mathbb{U}) \times_X J(\check{\mathbb{U}})$  is a pair of normal functions that determines the extension  $\mathbb{B}/\mathbb{Z}(1)$  of  $\mathbb{Z}_X(0)$  by  $\mathbb{U}$  and the extension  $W_{-1}\mathbb{B}$  of  $\mathbb{U}$  by  $\mathbb{Z}_X(1)$ . The biextension line bundle has a canonical metric  $|\cdot|_{\mathcal{B}}$ . A biextension  $\beta$  thus determines the real-valued function  $\log |\beta|_{\mathcal{B}} : X \rightarrow \mathbb{R}$ .

The pairing  $S$  also defines a morphism  $\mathbb{U} \rightarrow \check{\mathbb{U}}$  of variations of Hodge structure over  $X$ , and therefore a map  $i_S : J(\mathbb{U}) \rightarrow J(\check{\mathbb{U}})$ . Pulling back the line bundle  $\mathcal{B}$  along the map

$$(\text{id}, i_S) : J(\mathbb{U}) \rightarrow J(\mathbb{U}) \times_X J(\check{\mathbb{U}})$$

we obtain a metrized line bundle  $\hat{\mathcal{B}} \rightarrow J(\mathbb{U})$ . By [23, Prop. 7.3], the curvature of  $\hat{\mathcal{B}}$  is the translation-invariant, parallel 2-form  $2\omega_S$  on  $J(\mathbb{U})$  that corresponds to the bilinear form  $2S$ . Points of the associated  $\mathbb{C}^*$ -bundle  $\hat{\mathcal{B}}^*$  correspond to “symmetric biextensions”.

Denote by  $\phi_S \in H^2(J(\mathbb{U}))$  the class of  $\omega_S$ . Since  $2\omega_S$  represents  $c_1(\hat{\mathcal{B}})$ , the class  $2\phi_S$  is integral.

Suppose now that  $X = \overline{X} - Y$ , where  $\overline{X}$  is smooth and  $Y$  is a subvariety. Each normal function section  $\nu$  of  $J(\mathbb{U})$  thus determines a metrized holomorphic line



bundle  $\nu^*\widehat{\mathcal{B}}$  over  $X$ . One can ask whether it extends as a metrized line bundle to  $\overline{X}$ . Lear's thesis [28] implies that a power extends to a *continuously* metrized holomorphic line bundle over  $X - Y^{\text{sing}}$ .

**Theorem 6.1** (Lear [28]). *If  $\dim X = 1$  and  $\nu$  is a normal function section of  $J(\mathbb{U}) \rightarrow X$ , then there exists an integer  $N \geq 1$  such that the metrized holomorphic line bundle  $\nu^*\widehat{\mathcal{B}}^{\otimes N}$  over  $X$  extends to a holomorphic line bundle over  $\overline{X}$  with a continuous metric. Moreover, if  $\beta$  is a biextension section defined over  $X$  that projects to  $\nu$ , and if  $\mathbb{D}$  is a disk in  $\overline{X}$  with holomorphic coordinate  $t$  that is centered at a point of  $\overline{X} - X$ , then there is a rational number  $p/q$ , which depends only on the monodromy about the origin of  $\mathbb{D}$ , such that*

$$(4) \quad \left| \log |\beta(t)|_{\mathcal{B}} - \frac{p}{q} \log |t| \right|$$

*is bounded in a neighbourhood of  $t = 0$ .*

Note that the continuity of the metric ensures that the extension is uniquely determined. Since  $\overline{X}$  is smooth, every line bundle over  $\overline{X} - Y^{\text{sing}}$  extends uniquely to a line bundle over  $\overline{X}$ . Lear's Theorem thus implies the following result in the case  $\dim X \geq 1$ .

**Corollary 6.2.** *If  $\nu$  is a normal function section of  $J(\mathbb{U}) \rightarrow X$ , then there exists an integer  $N \geq 1$  such that the metrized holomorphic line bundle  $\nu^*\widehat{\mathcal{B}}^{\otimes N}$  over  $X$  extends to a holomorphic line bundle  $\overline{\mathcal{B}}_{N,\nu}$  over  $\overline{X}$  and its metric extends continuously over  $\overline{X} - Y^{\text{sing}}$ .*

This result implies that the class  $\nu^*\phi_S \in H^2(X)$  of  $\nu^*\omega_S$  has a natural extension to a class in  $H^2(\overline{X})$ ; namely  $c_1(\overline{\mathcal{B}}_{N,\nu})/2N$ .

**Corollary 6.3.** *If  $\nu$  is a normal function section of  $J(\mathbb{U}) \rightarrow X$ , then the class  $\nu^*\phi_S$  has a natural extension to a class  $\widehat{\nu^*\phi_S} \in H^2(\overline{X})$ .*

Lear's Theorem implies that the multiplicity of each boundary divisor in  $\widehat{\nu^*\phi_S}$  is determined by the asymptotics (4) of the restriction of the biextension to a disk transverse to the boundary divisor.

The previous result suggests that  $\nu^*\omega_S$ , regarded as a current on  $\overline{X}$ , is a natural representative of  $\widehat{\nu^*\phi_S}$ .

**Conjecture 6.4.** *If  $X$  is a curve, then the 2-form  $\nu^*\omega_S$  is integrable on  $X$  and*

$$\int_X \nu^*\omega_S = \frac{1}{2N} \int_{\overline{X}} c_1(\overline{\mathcal{B}}_{N,\nu}).$$

It is known that, in general, the metric does not extend continuously over  $Y^{\text{sing}}$  due to the phenomenon of “height jumping” which we shall discuss in Section 14 and which has been explained by Brosnan and Pearlstein in [2].

## 7. POLARIZATIONS

Polarizations play an important and subtle (if sometimes neglected) role in Hodge theory due to their positivity properties.

**7.1. Polarizations.** A *polarization* on a Hodge structure  $H$  of weight  $k$  is a  $(-1)^k$  symmetric bilinear form  $S$  on  $H_{\mathbb{Q}}$  satisfying the Riemann-Hodge bilinear relations:

- (i)  $S(H^{p,q}, \overline{H^{r,s}}) = 0$  unless  $p = r$  and  $q = s$ ;
- (ii)  $i^{p-q}S(v, \overline{v}) > 0$  when  $v \in H^{p,q}$  and  $v \neq 0$ .

A bilinear form  $S$  is a *weak polarization* on  $H$  if it satisfies the first condition and the weaker version  $i^{p-q}S(v, \overline{v}) \geq 0$  for all  $v \in H^{p,q}$  of the second.

Suppose that  $Y$  is a smooth projective variety of dimension  $n$ . Denote the hyperplane class by  $w$ . For  $k \leq n$ , define a bilinear form  $S$  on  $H^k(Y)$  by

$$(5) \quad S(u, v) = \int_Y u \wedge v \wedge w^{n-k}.$$

This is a non-degenerate,  $(-1)^k$  symmetric bilinear form. However, it is not a polarization in general. The Riemann-Hodge bilinear relations imply that the restriction of  $(-1)^{k(k-1)/2}S$  to  $PH^k(Y)$ , the primitive part of  $H^k(Y)$ , is a polarization. These provide the principal examples of polarized Hodge structures.

A variation of Hodge structure  $\mathbb{V}$  over a base  $X$  is *polarized* by  $S$  if  $S$  is a flat bilinear form on the variation which restricts to a polarization on each fiber.

**7.2. Some polarized variations of Hodge structure over  $\mathcal{A}_g$ .** The variations  $\mathbb{H}$ ,  $\mathbb{L}$  and  $\mathbb{V}$  defined in Section 5.4 have natural polarizations. The Riemann bilinear relations imply that the variation  $\mathbb{H}$  over  $\mathcal{A}_g$  is polarized by the inner product  $S_H$  introduced in Section 5.4. The corresponding polarization is easily described on the pullback of  $\mathbb{H}$  to  $\mathcal{M}_g$ . In this case, the fiber  $H_C$  of  $\mathbb{H}$  over the moduli point  $[C]$  of a smooth projective curve  $C$  is  $H^1(C, \mathbb{Z}(1))$ . Under this isomorphism,  $S_H$  corresponds to the inner product

$$S(u, v) = \int_C u \wedge v.$$

on  $H^1(C)$ .

The intersection form  $S_H$  extends to the skew symmetric bilinear form

$$S_L : \mathbb{L} \otimes \mathbb{L} \rightarrow \mathbb{Z}(1)$$

defined by

$$S_L(x_1 \wedge x_2 \wedge x_3, y_1 \wedge y_2 \wedge y_3) = \det(S_H(x_i, y_j)).$$

Note that this is *not* dual to the inner product  $S$  on  $H^3(\text{Jac } C)(1)$  defined in equation (5) above as is easily seen by a direct computation.

Denote the fiber  $\Lambda^3 H_C$  of  $\mathbb{L}$  over  $[C]$  by  $L_C$  and the fiber of  $\mathbb{V}$  over  $[C]$  by  $V_C$ . Define  $c : L_C \rightarrow H_C$  by

$$(6) \quad c(x \wedge y \wedge z) = S_H(y, z)x + S_H(z, x)y + S_H(x, y)z.$$

Regard  $S_H$  as an element of  $\Lambda^2 H_L$ . The canonical projection  $p : L_C \rightarrow V_C$  has a canonical  $\text{Sp}_g$ -invariant splitting  $j$ . It is defined by

$$j(p(x \wedge y \wedge z)) = x \wedge y \wedge z - S_H \wedge c(x \wedge y \wedge z)/(g-1).$$

A skew symmetric bilinear form on  $\mathbb{V}$  can be defined by

$$S_V(u, v) = (g-1)S_L(j(u), j(v)).$$

This form is integral and primitive.

**Proposition 7.1.** *The variations  $(\mathbb{H}, S_H)$ ,  $(\mathbb{L}, S_L)$  and  $(\mathbb{V}, S_V)$  are polarized variations of Hodge structure over  $\mathcal{A}_g$ , as are their pullbacks to  $\mathcal{M}_{g,n}^c$ .*

*Proof.* We have already seen that  $(\mathbb{H}, S_H)$  is a polarized variation of Hodge structure. For the rest, it suffices to show that  $\Lambda^3 H_1(C)$  is polarized by  $S_L$ . To do this, choose a basis  $u_1, \dots, u_g$  of  $H^{-1,0}$  that is orthonormal under the positive definite hermitian inner product

$$(u, v) = i^{-1} S_H(u, \bar{v}).$$

Then, for example,

$$i^{-3-0} S_L(u_1 \wedge u_2 \wedge u_3, \bar{u}_1 \wedge \bar{u}_2, \bar{u}_3) = \det(i^{-1} S_H(u_j, \bar{u}_k)) = 1 > 0.$$

and

$$i^{-2-(-1)} S_L(u_1 \wedge u_2 \wedge \bar{u}_3, \bar{u}_1 \wedge \bar{u}_2, u_3) = -i^{-1} i^3 \det(i^{-1} S_H(u_j, \bar{u}_k)) = -i^2 = 1 > 0.$$

The remaining computations follow by taking complex conjugates.  $\square$

The contraction (6) induces a projection  $c : \mathbb{L} \rightarrow \mathbb{H}$  of variations of Hodge structure. The canonical quotient mapping  $p : \mathbb{L} \rightarrow \mathbb{V}$  is also a morphism of variation of Hodge structure. The polarizations  $S_H$  of  $\mathbb{H}$  and  $S_V$  of  $\mathbb{V}$  can be pulled back along these projections to obtain the invariant inner product  $c^* S_H + p^* S_V$  on  $\mathbb{L}$ . For later use, we record the following fact:

**Lemma 7.2** ([22, Prop. 18]). *If  $g \geq 2$ , then  $c^* S_H + p^* S_V = (g-1) S_L$ .*  $\square$

## 8. COHOMOLOGY CLASSES

By Lemma 4.4 each invariant inner product on a variation of MHS  $\mathbb{U}$  over a smooth variety  $T$  gives rise to a parallel, translation invariant 2-form  $\omega$  on the associated bundle of intermediate jacobians  $J(\mathbb{U})$  and a cohomology class  $\phi \in H^2(J(\mathbb{U}))$ .

The polarizations  $S_H$ ,  $S_L$  and  $S_V$  of the variations of Hodge structure  $\mathbb{H}$ ,  $\mathbb{L}$  and  $\mathbb{V}$  over  $\mathcal{A}_g$  therefore give rise to cohomology classes

$$\phi_H \in H^2(J(\mathbb{H})), \quad \phi_L \in H^2(J(\mathbb{L})) \text{ and } \phi_V \in H^2(J(\mathbb{V})).$$

Denote their parallel, canonical translation invariant representatives by  $\omega_H$ ,  $\omega_L$  and  $\omega_V$ .

Recall that  $J(\mathbb{H}) = \mathcal{J}_g^c$ , the universal jacobian over  $\mathcal{M}_g^c$  and that  $\eta_g \in H^{2g}(\mathcal{J}_g^c)$  denotes the Poincaré dual of the 0-section  $Z_g$  of  $\mathcal{J}_g^c$ . A standard and elementary computation shows that if  $C$  is a smooth projective curve of genus  $g$ , then

$$\int_{\text{Jac } C} \omega_H^g = g!$$

Combining this with Proposition 4.5 yields the following result, which can also be deduced from [43, Cor. 2.2].

**Proposition 8.1.** *The class  $\eta_g$  of the zero section of  $\mathcal{J}_g^c$  in  $H^{2g}(\mathcal{J}_g^c)$  is  $\phi_H^g/g!$*   $\square$

Denote the restriction of  $\mathcal{J}_g$  to  $\overline{\mathcal{M}}_g - \Delta_0^{\text{sing}}$  by  $\mathcal{J}'_g$ . Zucker's Theorem [44] implies that every normal function section  $\mu$  of  $\mathcal{J}_g$  defined over  $\mathcal{M}_{g,n}^c$  extends to a section (also denoted  $\mu$ ) of  $\mathcal{J}'_g$  defined over  $\overline{\mathcal{M}}_{g,n} - \Delta_0^{\text{sing}}$ .

**Proposition 8.2.** *The class  $\phi_H \in H^2(\mathcal{J}_g^c)$  extends naturally to a class  $\hat{\phi}_H \in H^2(\mathcal{J}'_g)$ . It is characterized by the property*

$$[e]^* \hat{\phi}_H = e^2 \hat{\phi}_H \text{ for all } e \in \mathbb{Z}$$

*and has the property that  $\widehat{\mu^* \phi_H} = \mu^* \hat{\phi}_H$  for all normal function sections  $\mu$  of  $\mathcal{J}_g^c$  defined over  $\mathcal{M}_{g,n}^c$ .*

*Sketch of Proof.* The first rational cohomology of the smooth finite orbi-covering  $\overline{\mathcal{M}}_{g-1,2}$  of  $\Delta_0$  vanishes. The Gysin sequence thus gives an exact sequence

$$\mathbb{Q}\delta_0 \rightarrow H^2(\mathcal{J}'_g) \rightarrow H^2(\mathcal{J}_g^c) \rightarrow 0$$

of rational cohomology. For each integer  $e > 1$ , the endomorphism  $[e]$  of  $\mathcal{J}'_g$  induces an action on this sequence. It acts trivially on the left-hand term. Since  $[e]^*\phi_H = e^2\phi_H$  in  $H^2(\mathcal{J}_g^c)$ , it follows that  $\phi_H$  has a unique lift  $\hat{\phi}_H$  to  $H^2(\mathcal{J}'_g)$  with the property that  $[e]^*\hat{\phi}_H = e^2\hat{\phi}_H$ . The restriction of  $\hat{\phi}_H$  to the zero section  $\overline{\mathcal{M}}_g - \Delta_0^{\text{sing}}$  of  $\mathcal{J}'_g$  is trivial as  $[e]$  preserves the zero section,  $[e]$  acts trivially on  $H^2(\overline{\mathcal{M}}_g - \Delta_0^{\text{sing}})$  and since  $[e]^*\hat{\phi}_H = e^2\hat{\phi}_H$ .

To complete the proof, we need several facts about biextension line bundles. Suppose that  $f : Y \rightarrow X$  is a morphism of smooth varieties and that  $\mathbb{U}$  is a VHS over  $X$  polarized by  $S$ . Then the constructions of [14] imply that one has a commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{B}}_Y & \xrightarrow{f_{\mathcal{B}}} & \widehat{\mathcal{B}}_X \\ \downarrow & & \downarrow \\ J(f^*\mathbb{U}) & \xrightarrow{J(f)} & J(\mathbb{U}) \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

of biextension line bundles, where  $f_{\mathcal{B}}$  is a morphism of metrized line bundles. This implies that  $J(f)^*\phi_S = \phi_{f^*S}$ . The next fact, which follows from Lear's Theorem, is that if  $X'$  and  $Y'$  are smooth varieties in which  $X$  and  $Y$  are Zariski dense, and where  $X' - X$  is a smooth divisor in  $X'$ , and if  $\delta$  is a normal function section of  $J(\mathbb{U}) \rightarrow X$ , then

$$(7) \quad (f^*\widehat{\delta})^*\widehat{\phi}_{f^*S} = f^*(\widehat{\delta^*}\widehat{\phi}_S) \in H^2(Y').$$

We now apply this with  $X = \mathcal{J}_g^c$ ,  $X' = \mathcal{J}'_g$ ,  $Y = \mathcal{M}_{g,n}^c$  and  $Y' = \overline{\mathcal{M}}_{g,n} - \Delta_0^{\text{sing}}$ . The variation  $\mathbb{U}$  is the standard variation  $\mathbb{H}$ , so that  $J(\mathbb{U}) = \mathcal{J}_g^c \times_{\mathcal{M}_g^c} \mathcal{J}_g^c$ . Important here is the fact that  $\mathcal{J}_g^c$  is an algebraic variety. The normal function  $\delta$  will be the diagonal section of  $\mathcal{J}_g^c \times_{\mathcal{M}_g^c} \mathcal{J}_g^c \rightarrow \mathcal{J}_g^c$ . Finally,  $f : Y \rightarrow X$  will be a normal function  $\mu : \mathcal{M}_{g,n}^c \rightarrow \mathcal{J}_g^c$ , which is a morphism as  $\mathcal{J}_g^c \rightarrow \mathcal{M}_g^c$  is a family of (semi)-abelian varieties:

$$\begin{array}{ccc} \mathcal{J}'_{g,n} & \xrightarrow{(\Delta, \mu)} & \mathcal{J}'_g \times_{\overline{\mathcal{M}}_g} \mathcal{J}'_g \\ \mu \uparrow \downarrow & & \pi_1 \downarrow \uparrow \Delta \\ \overline{\mathcal{M}}_{g,n} - \Delta_0^{\text{sing}} & \xrightarrow{\mu} & \mathcal{J}'_g \end{array}$$

The class  $\phi_S \in H^2(\mathcal{J}'_g \times_{\overline{\mathcal{M}}_g} \mathcal{J}'_g)$  is  $\pi_2^*\phi_H$ , where  $\pi_1$  and  $\pi_2$  are the two projections  $\mathcal{J}'_g \times_{\overline{\mathcal{M}}_g} \mathcal{J}'_g \rightarrow \mathcal{J}'_g$ . It is now a tautology that  $\widehat{\Delta^*}\widehat{\phi}_S = \hat{\phi}_H \in H^2(\mathcal{J}'_g)$ . This and the naturality statement (7) now imply that  $\widehat{\mu^*}\widehat{\phi}_H = \mu^*(\widehat{\Delta^*}\widehat{\phi}_S) = \mu^*\hat{\phi}_H$ .  $\square$

It is important to note that Proposition 8.1 does not hold over  $\overline{\mathcal{M}}_g$ . This is because the restriction of  $\hat{\phi}_H$  to the zero section vanishes, whereas the conormal bundle of the zero section is the Hodge bundle, whose top Chern class is non-trivial.

We shall also need the invariant inner product  $\Delta$  on  $\mathbb{H} \oplus \mathbb{H}$  that is defined by

$$S_{\Delta}((u_1, v_1), (u_2, v_2)) = S_H(u_1, v_2) - S_H(u_2, v_1)$$

Even though it preserves the Hodge filtration, it is not a weak polarization as can be seen by restricting it to the diagonal of  $H \oplus H$  (where it is positive) and anti-diagonal (where it is negative). Denote the associated cohomology class in  $H^2(\mathcal{J}(\mathbb{H} \oplus \mathbb{H}))$  by  $\phi_{\Delta}$ . The class  $\phi_{\Delta}$  extends naturally to a class  $\hat{\phi}_{\Delta} \in H^2(\mathcal{J}'_g \times_{\overline{\mathcal{M}}_g} \mathcal{J}'_g)$ . The proof is similar to that of Proposition 8.2 and is left to the reader.

**8.1. The classes  $\mathcal{K}^*\phi_H$ ,  $\nu^*\phi_V$ ,  $\tilde{\nu}^*\phi_L$  and  $(\mathcal{K} \times \mathcal{K})^*\phi_{\Delta}$ .** Pulling back the classes  $\phi_H$ ,  $\phi_V$ ,  $\phi_L$  and  $\phi_{\Delta}$  along the normal functions  $\mathcal{K}$ ,  $\nu$ ,  $\tilde{\nu}$  and the normal function section  $\mathcal{K} \times \mathcal{K}$  of

$$J(\mathbb{H} \oplus \mathbb{H}) = J(\mathbb{H}) \times_{\mathcal{M}_{g,2}} J(\mathbb{H}) \rightarrow \mathcal{M}_{g,2}$$

defined by

$$\mathcal{K} \times \mathcal{K} : [C; x_1, x_2] \mapsto (\mathcal{K}(x_1), \mathcal{K}(x_2)) \in \text{Jac}(C) \times \text{Jac}(C).$$

we obtain rational cohomology classes  $\mathcal{K}^*\phi_H \in H^2(\mathcal{M}_{g,1}^c)$ ,  $\nu^*\phi_V \in H^2(\mathcal{M}_g^c)$ ,  $\tilde{\nu}^*\phi_L \in H^2(\mathcal{M}_{g,1}^c)$ , and  $(\mathcal{K} \times \mathcal{K})^*\phi_{\Delta} \in H^2(\mathcal{M}_{g,2}^c)$ . Lear's Theorem implies (via Cor. 6.3) that they extend naturally to classes  $\widehat{\nu^*\phi_V} \in H^2(\overline{\mathcal{M}}_g)$ ,  $\widehat{\tilde{\nu}^*\phi_L} \in H^2(\overline{\mathcal{M}}_{g,1})$ , and  $(\widehat{\mathcal{K} \times \mathcal{K}})^*\phi_{\Delta} \in H^2(\overline{\mathcal{M}}_{g,1})$ ,  $H^2(\overline{\mathcal{M}}_g)$ ,  $H^2(\overline{\mathcal{M}}_{g,1})$ , and  $H^2(\overline{\mathcal{M}}_{g,2})$ , respectively.

**Proposition 8.3.** *If  $g \geq 2$ , then  $(g-1)\widehat{\tilde{\nu}^*\phi_L} = \widehat{\nu^*\phi_V} + \widehat{\mathcal{K}^*\phi_H} \in H^2(\overline{\mathcal{M}}_{g,1})$ .*

*Proof.* This follows immediately from Lemma 7.2 and the fact [36] that  $c \circ \tilde{\nu} = \mathcal{K}$  and  $p \circ \tilde{\nu} = \nu$ .  $\square$

Our next task is to compute each of these classes. First we need to fix notation for the natural classes in  $H^2(\overline{\mathcal{M}}_{g,n})$ .

## 9. DIVISOR CLASSES

Denote the set  $\{x_1, x_2, \dots, x_n\}$  of marked points by  $I$ . Denote the relative dualizing sheaf of the universal curve  $\pi : \mathcal{C} \rightarrow \overline{\mathcal{M}}_{g,n}$  over  $\overline{\mathcal{M}}_{g,n}$  by  $w$ . Its pushforward  $\pi_*w$  is locally free of rank  $g$ . Recall that  $\lambda_1$  denotes the first Chern class of the Hodge bundle  $\pi_*w$ . The classes  $\psi_j$ ,  $x_j \in I$  are defined by

$$\psi_j := x_j^*c_1(w).$$

When  $n = 1$ , we will denote  $\psi_1$  by  $\psi$ . Note that this definition is different from the standard definition. We use is as our  $\psi$  classes are natural with respect to the forgetful maps  $\overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ .

Each component of the boundary divisor of  $\mathcal{M}_{g,n}$  in  $\overline{\mathcal{M}}_{g,n}$  has as its generic point a stable  $n$ -pointed curve of genus  $g$  with exactly one node. These components are:

- $\Delta_0$ : the generic point is an irreducible, geometrically connected  $n$ -pointed curve with one node;
- $\Delta_0^P$ , where  $P$  is a subset of  $I$  with  $|P| \geq 2$ : the generic point is a reducible curve with two geometrically connected components joined at a single node, one of which has genus 0 — the points in  $P$  lie on the genus 0 component minus its node, the remaining points  $P^c := I - P$  lie on the other (genus  $g$ ) component minus the node;

- $\Delta_h^P$ , where  $0 < h < g$  and  $P \subseteq I$  (possibly empty): the generic point is a reducible curve with exactly one node and two geometrically connected irreducible components, one of genus  $h$ , the other of genus  $g-h$ ; the points in  $P$  lie on the genus  $h$  component minus the node, and the rest  $P^c$  lie on the other component minus the node. Note that  $\Delta_{g-h}^{P^c} = \Delta_h^P$ .

Denote the classes of the divisors

$$\Delta_0, \Delta_0^P \ (P \subseteq I, |P| \geq 2), \Delta_h^P \ (0 < h < g, P \subseteq I)$$

by

$$\delta_0, \delta_0^P \ (P \subseteq I, |P| \geq 2), \delta_h^P \ (0 < h < g, P \subseteq I),$$

respectively. It is well-known that the classes

$$\lambda_1, \psi_j \ (x_j \in I), \delta_0, \delta_0^P \ (P \subseteq I, |P| \geq 2), \delta_h^P \ (0 < h \leq g/2, P \subseteq I)$$

comprise a basis of  $H^2(\overline{\mathcal{M}}_{g,n})$ .

One also has the Miller-Morita-Mumford classes

$$\kappa_j := \pi_* c_1(w)^{j+1} \in H^{2j}(\overline{\mathcal{M}}_g)$$

which are defined for  $j \geq 1$ . It follows from Grothendieck-Riemann-Roch (cf. [34]) that  $\kappa_1 = 12\lambda_1 - \delta$ , where

$$\delta = \sum_{j=0}^{\lfloor g/2 \rfloor} \delta_j \in H^2(\overline{\mathcal{M}}_g).$$

Define  $\kappa_j, \delta \in H^\bullet(\overline{\mathcal{M}}_{g,n})$  to be the pullbacks of  $\kappa_j, \delta \in H^\bullet(\overline{\mathcal{M}}_g)$  under the natural morphisms  $\overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_g$ . We thus have the alternate basis

$$\kappa_1, \psi_j \ (x_j \in I), \delta_0, \delta_0^P \ (P \subseteq I, |P| \geq 2), \delta_h^P \ (1 \leq h \leq g/2, P \subseteq I)$$

of  $H^2(\overline{\mathcal{M}}_{g,n})$ .

*Remark 9.1.* All of these divisor classes can be regarded as classes in  $H^2(\overline{\mathcal{M}}_{g,n})$  or in  $CH^1(\overline{\mathcal{M}}_{g,n})$ . Note that these two groups are isomorphic, so that any relation between divisor classes that holds in cohomology also holds in the Chow ring.

## 10. FORMULAS FOR $\nu^*\phi_V$ AND $\mathcal{K}^*\phi_H$

The computation of  $F_{\mathbf{d}}^*\eta_g$  will be reduced to the computation of the pullbacks of the classes  $\phi_H$  and  $\phi_\Delta$  along the normal functions  $\mathcal{K}_j$  and  $\delta_{i,j}$ . In this section we compute these basic classes. The formulas reflect the structure of Torelli groups.

The Moriwaki divisor is the class  $\widehat{\nu^*\phi_V}$ :

**Theorem 10.1** ([23, Thm. 1.3]). *If  $g \geq 2$ , then*

$$2\widehat{\nu^*\phi_V} = (8g+4)\lambda_1 - g\delta_0 - 4 \sum_{h=1}^{\lfloor g/2 \rfloor} h(g-h)\delta_h \in H^2(\overline{\mathcal{M}}_g, \mathbb{Z}).$$

The case  $g \geq 3$  is [23, Thm. 1.3]. When  $g = 2$ ,  $\nu = 0$  and the result follows from Mumford's computation [34] that  $10\lambda_1 = \delta_0 + 2\delta_1$ . Suitably interpreted, it holds in genus 1 as  $12\lambda_1 = \delta_0$  in  $H^2(\overline{\mathcal{M}}_{1,1}, \mathbb{Z})$ .

Proposition 8.2 implies that  $\widehat{\mathcal{K}^*\phi_H} = \mathcal{K}^*\hat{\phi}_H$ .

**Theorem 10.2.** *If  $g \geq 2$ , then*

$$2\mathcal{K}^*\hat{\phi}_H = 4g(g-1)\psi - \kappa_1 - \sum_{h=1}^{g-1} (2h-1)^2 \delta_{g-h}^{\{x\}} \in H^2(\overline{\mathcal{M}}_{g,1}, \mathbb{Z}).$$

This result holds trivially in genus 1 as  $\mathcal{K} \equiv 0$  and because  $\kappa_1 = 12\lambda_1 - \delta_0 = 0$ .

*Sketch of Proof.* Since  $\kappa_1 = 12\lambda_1 - \delta$ ,

$$\kappa_1 + \sum_{h=1}^{g-1} (2h-1)^2 \delta_{g-h}^{\{x\}} = 12\lambda_1 - \delta_0 + \sum_{h=1}^{g-1} 4h(h-1) \delta_{g-h}^{\{x\}}.$$

So it suffices to prove that

$$2\mathcal{K}^*\hat{\phi}_H = 4g(g-1)\psi - 12\lambda_1 + \delta_0 - \sum_{h=1}^{g-1} 4h(h-1) \delta_{g-h}^{\{x\}}.$$

This formula, modulo boundary terms, was proved by Morita [30, (1.7)]. Another proof is given in [22, Thm. 1].

The coefficient of  $\delta_h^{\{x\}}$  is computed using the method of [23, §11]. The Torelli group  $T_g$  is replaced by the Torelli group  $T_{g,1}$  associated to a 1-pointed surface. Instead of taking  $V = \Lambda^3 H / (\theta \wedge H)$ , we take it to be  $H$ . The quadratic form  $q$  (which is denoted  $S_V$  in this paper) is replaced by  $c^*S_H$ , where  $c : \Lambda^3 H \rightarrow H$  is the contraction (6). We sketch the monodromy computation using the notation of [23, §11].

The coefficient of  $\delta_h^{\{x\}}$  is  $-\hat{\tau}(\sigma_h)$ , where  $\sigma_h$  is a Dehn twist about a separating simple closed curve that divides a pointed, genus  $g$  reference surface into a surface of genus  $h$  (that does not contain the point) and a surface of genus  $g-h$ , and where  $\hat{\tau}$  is a representation of  $T_{g,1}$  into the Heisenberg group associated to  $(H, S_H)$ .

There is a symplectic basis  $a_1, b_1, \dots, a_g, b_g$  of  $H_{\mathbb{Z}}$  such that  $a_1, b_1, \dots, a_h, b_h$  is a basis of  $H'$ , the first homology of the genus  $h$  subsurface. Set  $\omega' = a_1 \wedge b_1 + \dots + a_h \wedge b_h$ , the symplectic form of  $H'$ . If  $u \in H$ , then  $c(u \wedge \omega') = (h-1)u$ . Thus

$$\begin{aligned} \hat{\tau}_h(\sigma_h) &= \frac{8}{2h-2} \sum_{j=1}^h S_H(c(a_j \wedge \omega'), c(b_j \wedge \omega')) \\ &= \frac{8(h-1)^2}{2h-2} \sum_{j=1}^h S_H(a_j, b_j) = 4h(h-1). \end{aligned}$$

It remains to compute the coefficient of  $\delta_0$ . The most direct way to compute it is by restricting to a curve in the hyperelliptic locus. First note that if  $C$  is a hyperelliptic curve and  $x \in C$  is a Weierstrass point, then  $\mathcal{K}(C, x) = 0$ . Call such a pair  $(C, x)$  a *hyperelliptic pointed curve*. Suppose that  $T$  is a smooth, complete curve and that  $f : T \rightarrow \overline{\mathcal{M}}_{g,1}$  is a morphism where  $f(t)$  is the moduli point of an irreducible hyperelliptic pointed curve for each  $t \in T$ .<sup>8</sup> The normal function  $f^*\mathcal{K}$

<sup>8</sup>For example, we can take  $T = \mathbb{P}^1$  and  $f$  the morphism associated to the family

$$v^2 = (u-t)u \prod_{j=1}^{2g} (u-a_j),$$

where  $t \in \mathbb{C}$  and the  $a_j$  are distinct non-zero complex numbers. A section of Weierstrass points is given by  $x = (0, 0)$ .

vanishes identically on  $T$ , which implies that  $f^*\mathcal{K}^*\widehat{\mathcal{B}}$  is trivial as a metrized line bundle over  $T - f^{-1}\Delta_0$ . Its extension as a metrized line bundle to  $T$  is therefore trivial. This implies the vanishing of

$$f^*\mathcal{K}^*\hat{\phi}_H \in H^2(T).$$

On the other hand, standard techniques can be used to show that

$$f^*(8\lambda_1 + 4g\psi - \delta_0) = 0.$$

The Cornalba-Harris relation [6, Prop. 4.7] implies that

$$f^*((8g + 4)\lambda_1 - g\delta_0) = 0 \in \text{Pic } T.$$

It follows that

$$f^*(4g(g-1)\psi - 12\lambda_1 + \delta_0) = (g-1)f^*(8\lambda_1 + 4g\psi - \delta_0) - f^*((8g+4)\lambda_1 - g\delta_0) = 0.$$

These two facts together imply that the coefficient of  $\delta_0$  in  $\mathcal{K}^*\hat{\phi}_H$  is 1.  $\square$

Since  $\kappa_1 = 12\lambda_1 - \delta$ , Theorems 10.1 and 10.2 and Proposition 8.3 imply the following result when  $g \geq 2$ . The case  $g = 1$  follows from the fact that  $\psi = \lambda_1$  and the well-known relation  $\delta_0 = 12\lambda_1$  in  $\text{Pic } \overline{\mathcal{M}}_{1,1}$ .

An immediate consequence of Lemma 7.2 and the two previous results is the following formula for  $\widehat{\tilde{\nu}^*\phi_L}$ .

**Corollary 10.3.** *For all  $g \geq 1$ ,*

$$2\tilde{\nu}^*\phi_L = 8\lambda_1 + 4g\psi - \delta_0 - 4 \sum_{h=1}^{g-1} h\delta_{g-h}^{\{x\}} \in H^2(\overline{\mathcal{M}}_{g,1}, \mathbb{Z}).$$

The next result is needed in the solution of Eliashberg's problem.

**Theorem 10.4.** *If  $g \geq 2$ , then in  $H^2(\overline{\mathcal{M}}_{g,2}, \mathbb{Z})$  we have*

$$\begin{aligned} (\mathcal{K} \times \mathcal{K})^*\hat{\phi}_\Delta &= (2g-2)(\psi_1 + \psi_2) - \kappa_1 - (2g-2)^2\delta_0^{\{x_1, x_2\}} \\ &\quad - \sum_{h=1}^{g-1} (2h-1)^2 \delta_{g-h}^{\{x_1, x_2\}} + (2h-1)(2(g-h)-1)(\delta_h^{\{x_1\}} + \delta_{g-h}^{\{x_2\}})/2. \end{aligned}$$

Note that in this and subsequent formulas, we will often sum from  $h = 1$  to  $h = g-1$  and over all subsets  $P$  of  $I$ . Because of this, some terms will appear twice as  $\delta_h^P = \delta_{g-h}^P$ . We do this to emphasize the symmetry of the formulas and to facilitate later computations.

*Proof.* Modulo the coefficients of the  $\delta_h^{\{x_1\}}$ , this formula can be computed

- (i) by restricting  $(\mathcal{K} \times \mathcal{K})^*\phi_\Delta$  to any fiber  $C \times C$  and
- (ii) from Theorem 10.2 by restricting to the diagonal  $\overline{\mathcal{M}}_{g,1} \rightarrow \overline{\mathcal{M}}_{g,2}$ , noting that the restriction of  $\hat{\phi}_\Delta$  to the diagonal  $\mathcal{J}_g$  of  $\mathcal{J}_g \times_{\mathcal{M}_{g,2}^c} \mathcal{J}_g$  is  $2\hat{\phi}_H$ .

These computations are straightforward, once one notes that the divisor  $\Delta_0^{\{x_1, x_2\}}$  is the “diagonal”  $\mathcal{M}_{g,1}^c \rightarrow \mathcal{M}_{g,2}^c$  in  $\mathcal{M}_{g,2}^c$  and that the Chern class of its normal



bundle is  $\psi$ . If we restrict to a single curve  $C$ , then in  $H^2(C \times C)$

$$\begin{aligned} (\mathcal{K} \times \mathcal{K})^* \hat{\phi}_\Delta &= (2g-2)^2 \sum_{j=1}^g (a_j^{(1)} \wedge b_j^{(2)} - b_j^{(1)} \wedge a_j^{(2)}) \\ &= (2g-2)^2 ([\text{point}]^{(1)} + [\text{point}]^{(2)} - [\text{diagonal}]) \\ &= (2g-2)(\psi_1 + \psi_2) - (2g-2)^2 \delta_0^{\{x_1, x_2\}}. \end{aligned}$$

Here  $a_1, \dots, b_g$  is a symplectic basis of  $H_1(C)$  and, for  $x \in H^1(C)$ ,  $x^{(k)}$  denotes the pullback of  $x$  under the  $k$ th projection  $p_k : C^2 \rightarrow C$ .

It remains to compute the coefficient of  $\delta_h^{x_1}$  when  $0 < h < g$ . This we do using a test curve suggested by Sam Grushevsky. Since  $\hat{\phi}_\Delta$  is invariant when the two factors of  $\mathcal{J}_g^c \times_{\mathcal{M}_{g,2}^c} \mathcal{J}_g^c$  are swapped, it follows that the formula for  $(\mathcal{K} \times \mathcal{K})^* \hat{\phi}_\Delta$  is symmetric in  $x_1$  and  $x_2$ . Since  $\delta_h^{\{x_1\}} = \delta_{g-h}^{\{x_2\}}$ , the formula is also invariant when  $h$  is replaced by  $g-h$ . We therefore conclude that

$$\begin{aligned} (8) \quad (\mathcal{K} \times \mathcal{K})^* \hat{\phi}_\Delta &= (2g-2)(\psi_1 + \psi_2) - \kappa_1 - (2g-2)^2 \delta_0^{\{x_1, x_2\}} \\ &\quad - \sum_{h=1}^{g-1} (2h-1)^2 \delta_{g-h}^{\{x_1, x_2\}} + \sum_{h=1}^{g-1} c_h \delta_h^{\{x_1\}} \end{aligned}$$

where  $c_h = c_{g-h}$ .

Suppose that  $0 < h < g$ . Fix pointed smooth projective curves  $(C', P')$  and  $(C'', P'')$  with  $g(C') = h$  and  $g(C'') = g-h$ . Let  $C$  be the nodal genus  $g$  curve with three components  $C'$ ,  $C''$  and  $\mathbb{P}^1$ , where  $C'$  is attached to  $\mathbb{P}^1$  by identifying  $P' \in C'$  with  $0 \in \mathbb{P}^1$  and  $P'' \in C''$  with  $\infty \in \mathbb{P}^1$ . For  $t \in \mathbb{P}^1 - \{0, 1, \infty\}$  let  $C_t$  be the stable 2-pointed curve  $(C; 1, t)$ . The closure  $T$  of the curve

$$\mathbb{P}^1 - \{0, 1, \infty\} \rightarrow \mathcal{M}_{g,2}^c, \quad t \mapsto [C_t]$$

is a copy of  $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$  imbedded in  $\mathcal{M}_{g,2}^c$ .

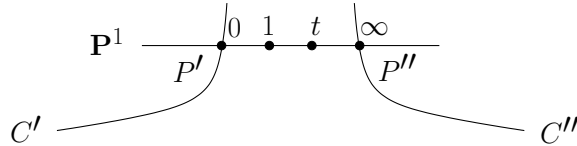


FIGURE 1. The 2-pointed curve  $C_t$

The restriction of  $\mathcal{K} \times \mathcal{K}$  to  $T$  takes the constant value

$$((2h-2)P' - K_{C'}, (2(g-h)-2)P'' - K_{C''}) \in \text{Jac } C' \times \text{Jac } C'' \cong \text{Jac } C,$$

which implies that  $(\mathcal{K} \times \mathcal{K})^* \phi_\Delta = 0$ . The coefficient  $c_h$  is computed by evaluating the right hand side (RHS) of (8) on  $T$ .

The curve  $T$  is contained in  $\Delta_h^{\{x_1, x_2\}} \cap \Delta_{g-h}^{\{x_1, x_2\}}$  and intersects the three boundary divisors  $\Delta_0^{\{x_1, x_2\}}$ ,  $\Delta_h^{\{x_1\}}$  and  $\Delta_h^{\{x_2\}}$  transversely in three distinct points. These are the three boundary points of  $\overline{\mathcal{M}}_{0,4} \cong T$ . It does not intersect any other boundary divisors. Consequently,

$$\int_T \delta_0^{\{x_1, x_2\}} = \int_T \delta_h^{\{x_1\}} = \int_T \delta_h^{\{x_2\}} = 1.$$

The projection formula can be used to evaluate the other terms of the RHS of (8) on  $T$ . Let

$$q : \overline{\mathcal{M}}_{g,2} \rightarrow \overline{\mathcal{M}}_g, \quad p_j : \overline{\mathcal{M}}_{g,2} \rightarrow \overline{\mathcal{M}}_{g,1} \quad j = 1, 2$$

denote the natural projections, where  $p_j([C; x_1, x_2]) = [C, x_j]$ . Note that  $q$  and the  $p_j$  collapse  $T$  to a point. Since  $\psi_j = p_j^* \psi$  and  $\kappa_1 = q^* \kappa_1$ , the projection formula implies that

$$\int_T \kappa_1 = \int_T q^* \kappa_1 = \int_{q_* T} \kappa_1 = 0 \quad \text{and} \quad \int_T \psi_j = \int_T p_j^* \psi = \int_{p_j * T} \psi = 0.$$

Since  $p_1^* \delta_h^{\{x\}} = \delta_h^{\{x_1, x_2\}} + \delta_h^{\{x_1\}}$ , the projection formula implies that

$$\int_T \delta_h^{\{x_1, x_2\}} = - \int_T \delta_h^{\{x_1\}} = -1.$$

Similarly,  $\int_T \delta_{g-h}^{\{x_1, x_2\}} = -1$ .

Evaluating the expression (8) on  $T$  we obtain

$$0 = 0 + 0 - (2g - 2)^2 + (2h - 1)^2 + (2(g - h) - 1)^2 + c_h + c_{g-h}.$$

Since  $c_h = c_{g-h}$ , this implies that  $c_h = (2h - 1)(2(g - h) - 1)$ .  $\square$

#### 11. SOLUTION OF ELIASHBERG'S PROBLEM OVER $\mathcal{M}_{g,n}^c$ WHEN $g > 1$

In this section, we solve Eliashberg's problem over  $\mathcal{M}_{g,n}^c$  when  $g > 2$ . A complete solution in the genus 1 case is given in the following section. The solution in genus  $> 1$  is a direct consequence of Proposition 8.1 and Theorem 11.5 below. Related work on Eliashberg's problem has been obtained independently by Cavalieri and Marcus [5] via Gromov-Witten theory.

Fix an integral vector  $\mathbf{d} = (d_1, \dots, d_n)$  with  $\sum_j d_j = 0$ . As in the introduction, we have the section

$$F_{\mathbf{d}} : \mathcal{M}_{g,n}^c \rightarrow \mathcal{J}_g$$

of the universal jacobian defined by

$$F_{\mathbf{d}} : (x_1, \dots, x_d) \mapsto \left[ \sum_{j=1}^n d_j x_j \right] \in \text{Jac } C.$$

For each subset  $P$  of  $\{x_1, \dots, x_n\}$ , set  $d_P = \sum_{j \in P} d_j$ . Since  $d_P + d_{P^c} = 0$ ,  $d_P^2 \delta_h^P = d_{P^c}^2 \delta_{g-h}^{P^c}$ .

**Theorem 11.1.** *If  $g \geq 2$ , then in  $H^{2g}(\mathcal{M}_{g,n}^c)$  we have*

$$F_{\mathbf{d}}^* \eta_g = \frac{1}{g!} \left( \sum_{j=1}^n d_j^2 \psi_j / 2 - \sum_{P \subseteq I} \sum_{\{x_j, x_k\} \subseteq P} d_j d_k \delta_0^P - \frac{1}{4} \sum_{P \subseteq I} \sum_{h=1}^{g-1} d_P^2 \delta_h^P \right)^g.$$

Recall that our definition of the  $\psi_j$  differs from the commonly used one. Here  $\psi_j := x_j^* c_1(\omega)$ , where  $\omega$  is the relative dualizing sheaf of the universal curve. Note, too, that since  $\delta_h^P = \delta_h^{P^c}$  and since we are summing from  $h = 1$  to  $h = g - 1$  in this and other results in this section, some boundary divisors occur twice in this expression.

I do not know if this formula holds in the Chow ring. Although this formula makes sense in  $H^{2g}(\overline{\mathcal{M}}_{g,n} - \Delta_0^{\text{sing}})$ , it does not hold there. For example, when

$\mathbf{d} = 0$  the left hand side is the non-trivial class  $(-1)^g \lambda_g$ , whereas the right-hand side vanishes. A result of Ekedahl and van der Geer [9] implies that, in  $CH^g(\overline{\mathcal{M}}_g - \Delta_0^{\text{sing}})$ ,  $\lambda_g$  is  $(-1)^g \zeta(1-2g)$  times a natural class, where  $\zeta(s)$  denotes the Riemann zeta function. This suggests that the class

$$F_{\mathbf{d}}^* \eta_g - \frac{1}{g!} (F_{\mathbf{d}}^* \hat{\phi}_H)^g \in CH^g(\overline{\mathcal{M}}_{g,n} - \Delta_0^{\text{sing}})$$

should be interesting. In particular, does Proposition 8.1 hold in  $CH^g(\mathcal{J}_g^c)$ ? This makes sense, as  $\phi_H = \theta - \lambda_1/2$  in  $\text{Pic } \mathcal{J}_g^c$ . (Cf. the proof of Theorem 11.6 below.)

**11.1. The approach and reduction.** Denote the pullback of  $\mathcal{J}_g$  to  $\overline{\mathcal{M}}_{g,n}$  by  $\mathcal{J}_{g,n}$  and its restriction to  $\mathcal{M}_{g,n}^c$  by  $\mathcal{J}_{g,n}^c$ . We will consider a more general situation. Namely, we'll assume that  $\mathbf{d} \in \mathbb{Z}^n$  and that  $m \in \mathbb{Z}$  satisfy

$$\sum_{j=1}^n d_j = (2g-2)m.$$

Then one has the section

$$F_{\mathbf{d}} : [C; x_1, \dots, x_n] \mapsto \sum_{j=1}^n d_j x_j - m K_C \in \text{Jac } C$$

of  $\mathcal{J}_{g,n}^c$  over  $\mathcal{M}_{g,n}^c$ . Our goal is to compute  $F_{\mathbf{d}}^* \phi_H$ .  
Set

$$\mathcal{J}_{g,n}^m := \underbrace{\mathcal{J}_{g,n} \times_{\overline{\mathcal{M}}_{g,n}} \mathcal{J}_{g,n} \times_{\overline{\mathcal{M}}_{g,n}} \cdots \times_{\overline{\mathcal{M}}_{g,n}} \mathcal{J}_{g,n}}_m.$$

Denote its restriction to  $\mathcal{M}_{g,n}^c$  by  $\mathcal{J}_{g,n}^{c,m}$ . Let

$$\mathcal{K}_n : \overline{\mathcal{M}}_{g,n} - \Delta_0^{\text{sing}} \rightarrow \mathcal{J}_{g,n}^n$$

be the  $n$ th power of  $\mathcal{K}$  — that is, the section of  $\mathcal{J}_{g,n}^{c,n} \rightarrow \mathcal{M}_{g,n}^c$  defined by

$$\mathcal{K}_n : (C; x_1, \dots, x_n) \mapsto (\mathcal{K}(x_1), \mathcal{K}(x_2), \dots, \mathcal{K}(x_n)) \in (\text{Jac } C)^n.$$

Note that  $\mathcal{K}_2 = \mathcal{K} \times \mathcal{K}$ .

**Proposition 11.2.** *Define  $\mathbf{d} : \mathcal{J}_{g,n}^n \rightarrow \mathcal{J}_{g,n}^n$  by*

$$\mathbf{d} : (u_1, \dots, u_n) \mapsto (d_1 u_1, \dots, d_n u_n)$$

*and  $\text{tr}_n : \mathcal{J}_{g,n}^n \rightarrow \mathcal{J}_{g,n}$  by*

$$\text{tr}_n : (u_1, \dots, u_n) \mapsto u_1 + \cdots + u_n.$$

*Then the mapping*

$$\overline{\mathcal{M}}_{g,n} - \Delta_0^{\text{sing}} \xrightarrow{\mathcal{K}_n} \mathcal{J}_{g,n}^n \xrightarrow{\mathbf{d}} \mathcal{J}_{g,n}^n \xrightarrow{\text{tr}_n} \mathcal{J}_{g,n}$$

*equals  $(2g-2)F_{\mathbf{d}}$ .* □

This formula allows the reduction of the computation of  $F_{\mathbf{d}}^* \phi_H$  to more basic computations. Denote the pullback of  $\phi_H$  under the  $j$ th projection

$$p_j : \mathcal{J}_{g,n}^{c,n} \rightarrow \mathcal{J}_{g,n}^c$$

by  $\phi_{j,j}$ . For  $j \neq k$ , denote the pullback of  $\phi_{\Delta}$  under the  $(j,k)$ th projection

$$p_{j,k} : \mathcal{J}_{g,n}^{c,n} \rightarrow \mathcal{J}_{g,n}^{c,2} \quad (u_1, \dots, u_n) \mapsto (u_j, u_k)$$

by  $\phi_{j,k}$ .

**Lemma 11.3.** *With notation as above,*

$$\mathbf{d}^* \phi_{j,k} = d_j d_k \phi_{j,k} \text{ and } \mathrm{tr}_n^* \phi_H = \sum_{j \leq k} \phi_{j,k}.$$

*Proof.* Since all of the classes  $\phi_{j,k}$  are represented by parallel, translation invariant forms, to prove the result, it suffices to prove the result in the cohomology of the jacobian  $J := \mathrm{Jac} C$  of a single smooth projective curve  $C$ .

Set  $J = \mathrm{Jac} C$ . Note that the ring homomorphism

$$H^\bullet(J) \rightarrow H^\bullet(J)^{\otimes n}$$

induced by the addition map  $J^n \rightarrow J$  is, in degree 1, given by

$$x \mapsto \sum_{\substack{a+b=n-1 \\ a,b \geq 0}} 1^{\otimes a} \otimes x \otimes 1^{\otimes b}.$$

The formula for  $\mathrm{tr}_n^*$  follows using the fact that this is a ring homomorphism.

The formula for  $\mathbf{d}^*$  follows as the map  $[e] : J \rightarrow J$  is multiplication by  $e$  on  $H^1(J)$ , and therefore multiplication by  $e^k$  on  $H^k(J)$ .  $\square$

Recall that  $\phi_H \in H^2(\mathcal{J}_g^c)$  and  $\phi_\Delta \in H^2(\mathcal{J}_g \times_{\mathcal{M}_g^c} \mathcal{J}_g)$  extend naturally to classes

$$\hat{\phi}_H \in H^2(\mathcal{J}'_g) \text{ and } \hat{\phi}_\Delta \in H^2(\mathcal{J}'_g \times_{\overline{\mathcal{M}}_g} \mathcal{J}'_g),$$

where  $\mathcal{J}'_g$  denotes the universal jacobian over  $\overline{\mathcal{M}}_g - \Delta_0^{\mathrm{sing}}$ . Using the scaling by the action of  $(e_1, \dots, e_n) \in \mathbb{Z}^n$  on the cohomology of the  $n$ th power of  $\mathcal{J}'_g \rightarrow \overline{\mathcal{M}}_g - \Delta_0^{\mathrm{sing}}$  one can show that Lemma 11.3 holds when  $\phi_H$  and  $\phi_\Delta$  are replaced by the extended classes  $\hat{\phi}_H$  and  $\hat{\phi}_\Delta$ . We therefore have:

**Corollary 11.4.** *If  $g \geq 2$ , then*

$$(2g-2)^2 F_{\mathbf{d}}^* \hat{\phi}_H = \sum_{j=1}^n d_j^2 \pi_j^* \mathcal{K}^* \hat{\phi}_H + \sum_{1 \leq j < k \leq n} d_j d_k \pi_{j,k}^* (\mathcal{K} \times \mathcal{K})^* \hat{\phi}_\Delta \in H^2(\mathcal{M}_{g,n}^c),$$

where  $\pi_j : \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,1}$  and  $\pi_{j,k} : \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,2}$  denote the natural projections.

The following result is the main computation of this section.

**Theorem 11.5.** *If  $g \geq 2$  and  $\mathbf{d} \in \mathbb{Z}^n$  and  $m \in \mathbb{Z}$  satisfy  $\sum d_j = (2g-2)m$ , then in  $H^2(\overline{\mathcal{M}}_{g,n})$  we have*

$$\begin{aligned} F_{\mathbf{d}}^* \hat{\phi}_H = & -m^2 \kappa_1 / 2 + \sum_{j=1}^n (d_j m + d_j^2 / 2) \psi_j - \sum_{P \subseteq I} \sum_{\{x_j, x_k\} \subseteq P} d_j d_k \delta_0^P \\ & - \frac{1}{4} \sum_{P \subseteq I} \sum_{h=1}^{g-1} (d_P - (2h-1)m)^2 \delta_h^P. \end{aligned}$$

Recall that  $\delta_h^P = \delta_{g-h}^{P^c}$ . Note that, in this expression, the coefficients of  $\delta_h^P$  and  $\delta_{g-h}^{P^c}$  are equal as  $d_P + d_{P^c} = (2g-2)m$ . Note, too, that one recovers Theorem 10.2 when  $n = m = 1$  and  $d_1 = 2g-2$ .

Theorem 11.1 follows directly from Proposition 8.1 and the case  $m = 0$ . The corresponding genus 1 statement is proved in the following section. The proof below fails in genus 1 as we cannot divide by  $2g-2$ .

*Proof.* This follows from Theorems 10.2 and 10.4 and Corollary 11.4. First observe that the coefficient of  $\kappa_1$  in  $(2g-2)^2 F_{\mathbf{d}}^* \hat{\phi}_H$  is

$$-\sum_j d_j^2/2 - \sum_{j < k} d_j d_k = -(d_1 + \cdots + d_n)^2/2 = -(2g-2)^2 m^2/2.$$

Next, the coefficient of  $\psi_j$  in  $(2g-2)^2 F_{\mathbf{d}}^* \hat{\phi}_H$  is

$$\begin{aligned} (2g-2) \sum_{k \neq j} d_j d_k + \frac{1}{2} 4g(g-1) d_j^2 &= (2g-2) [(d_1 + \cdots + d_n - d_j) d_j + g d_j^2] \\ &= (2g-2)^2 d_j m + (2g-2)(g-1) d_j^2 \\ &= (2g-2)^2 (d_j m + d_j^2/2). \end{aligned}$$

Since

$$\pi_{j,k}^* \delta_0^{\{x_1, x_2\}} = \sum_{\{x_j, x_k\} \subseteq P} \delta_0^P, \quad j \neq k$$

the coefficient of  $\delta_0^P$  in the expression for  $(2g-2)^2 F_{\mathbf{d}}^* \hat{\phi}_H$  is

$$-(2g-2)^2 \sum_{\{x_j, x_k\} \subseteq P} d_j d_k \delta_0^P.$$

When  $1 \leq h \leq g-1$  and  $j \neq k$

$$\pi_j^* \delta_h^{\{x\}} = \sum_{x_j \in P} \delta_h^P, \quad \pi_{j,k}^* \delta_h^{\{x_1, x_2\}} = \sum_{\{x_j, x_k\} \subseteq P} \delta_h^P, \quad \text{and} \quad \pi_{j,k}^* \delta_h^{x_1} = \sum_{\substack{x_j \in P \\ x_k \in P^c}} \delta_h^P.$$

Then, computing formally, we see that the coefficient of  $\delta_h^P$  in  $(2g-2)^2 F_{\mathbf{d}}^* \hat{\phi}_H$  is the coefficient of  $\delta_h^P$  in

$$\begin{aligned} &-(2(g-h)-1)^2 \sum_j d_j^2 \pi_j^* \delta_h^{\{x\}}/2 - (2(g-h)-1)^2 \sum_{j < k} d_j d_k \pi_{j,k}^* \delta_h^{\{x_1, x_2\}} \\ &\quad + (2h-1)(2(g-h)-1) \sum_{j < k} d_j d_k \pi_{j,k}^* (\delta_h^{x_1} + \delta_{g-h}^{x_2})/2, \end{aligned}$$

which is

$$\begin{aligned} &-\frac{(2(g-h)-1)^2}{2} \left( \sum_{x_j \in P} d_j^2 + \sum_{\substack{x_j, x_k \in P \\ x_j \neq x_k}} d_j d_k \right) + \frac{(2h-1)(2(g-h)-1)}{2} \sum_{\substack{x_j \in P \\ x_k \in P^c}} d_j d_k \\ &= -\frac{1}{2} \left( (2(g-h)-1)^2 d_P^2 - (2(g-h)-1)(2h-1) d_P d_{P^c} \right). \end{aligned}$$

Since  $\delta_h^P = \delta_{g-h}^{P^c}$ , the coefficient of  $\delta_h^P$  can be chosen (and will be chosen) to be the average of the formally computed coefficients of  $\delta_h^P$  and  $\delta_{g-h}^{P^c}$ , which is

$$\begin{aligned} &-\frac{1}{4} \left( (2(g-h)-1)^2 d_P^2 - (2(g-h)-1)(2h-1) d_P d_{P^c} + (2h-1)^2 d_{P^c}^2 \right) \\ &= -\frac{1}{4} \left( (2(g-h)-1) d_P - (2h-1) d_{P^c} \right)^2 \\ &= -\frac{1}{4} \left( (2(g-h)-1) d_P + (2h-1) (d_P - (2g-2)m) \right)^2 \\ &= -\frac{(2g-2)^2}{4} (d_P - (2h-1)m)^2 \end{aligned}$$

as  $d_P + d_{P^c} = (2g - 2)m$ .  $\square$

**11.2. Variants.** Theorem 11.5 can be adapted to establish more general results. Suppose that  $r|2g - 2$  and suppose that  $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{Z}^n$  and  $m \in \mathbb{Z}$  satisfy

$$\sum_{j=1}^n d_j = m(2g - 2)/r.$$

Let  $f : X \rightarrow \overline{\mathcal{M}}_{g,n} - \Delta_0^{\text{sing}}$  be a morphism over which there is a globally defined  $r$ th root  $\alpha$  of the relative dualizing sheaf. Then one has the section  $F_{\mathbf{d}}$  of  $\mathcal{J}_{g,n}^c$  over  $X$  defined by

$$E_{\mathbf{d}} : [C; x_1, \dots, x_n] \mapsto -m\alpha + \sum_{j=1}^n d_j x_j \in \text{Jac } C.$$

As above,  $d_P = \sum_{x_j \in P} d_j$ .

**Theorem 11.6.** *The class  $E_{\mathbf{d}}^* \hat{\phi}_H \in H^2(X)$  is the pullback along  $f$  of the class*

$$\begin{aligned} & - (m/r)^2 \kappa_1 / 2 + \sum_{j=1}^n (d_j(m/r) + d_j^2/2) \psi_j - \sum_{P \subseteq I} \sum_{\{x_j, x_k\} \subseteq P} d_j d_k \delta_0^P \\ & - \frac{1}{4} \sum_{P \subseteq I} \sum_{h=1}^{g-1} (d_P - (2h-1)(m/r))^2 \delta_h^P \in H^2(\overline{\mathcal{M}}_{g,n}). \end{aligned}$$

*Proof.* Since the diagram

$$\begin{array}{ccc} & & \mathcal{J}_{g,n} \\ & \nearrow rE_{\mathbf{d}} & \downarrow F_{r\mathbf{d}} \\ X & \xrightarrow{f} & \overline{\mathcal{M}}_{g,n} - \Delta_0^{\text{sing}} \end{array}$$

commutes (i.e.,  $F_{r\mathbf{d}} \circ f = rE_{\mathbf{d}}$ ), and since the extended class  $\hat{\phi}_H$  satisfies  $[e]^* \hat{\phi}_H = e^2 \hat{\phi}_H$ , we have

$$E_{\mathbf{d}}^* \hat{\phi}_H = r^{-2} E_{r\mathbf{d}} \hat{\phi}_H = r^{-2} f^* F_{r\mathbf{d}} \hat{\phi}_H.$$

The result now follows from Theorem 11.5.  $\square$

This result can be used to give a partial solution to a problem posed to me by Joe Harris. Suppose that  $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{Z}^n$  satisfies  $\sum_{j=1}^n d_j = g - 1$ . Then one has the section

$$G_{\mathbf{d}} : [C; x_1, \dots, x_n] \mapsto \sum_j d_j x_j \in \text{Pic}^{g-1} C$$

of the relative Picard bundle

$$\mathcal{P}_{g,n} := \text{Pic}_{\mathcal{C}_{g,n}/\mathcal{M}_{g,n}}^{g-1}$$

over  $\mathcal{M}_{g,n}$ . This contains the divisor  $W$  of effective divisor classes of degree  $g - 1$ . The pullback  $G_{\mathbf{d}}^* W$  is the divisor in  $\mathcal{M}_{g,n}$  consisting of those  $[C; x_1, \dots, x_n]$  where  $h^0(C, \sum d_j x_j) > 0$ . Denote its closure in  $\overline{\mathcal{M}}_{g,n}$  by  $W_{\mathbf{d}}$ . Harris' problem is to compute the class of  $W_{\mathbf{d}}$  in terms of standard classes. This is a subtle problem as  $\mathcal{P}_{g,n}$  is not separated over  $\mathcal{M}_{g,n}^c$ .

Although we cannot solve this problem, we can solve the following closely related problem. Let  $X \rightarrow \overline{\mathcal{M}}_{g,n} - \Delta_0^{\text{sing}}$  be any dominant morphism on which there is a globally defined theta characteristic  $\alpha$ . Denote the inverse image of  $\mathcal{M}_{g,n}^c$  in  $X$  by  $X^c$  and the inverse image in  $X$  of  $\mathcal{M}_{g,n}$  by  $X^\circ$ . Denote the universal jacobian over  $X$  by  $\mathcal{J}_X$  and its restrictions to  $X^c$  and  $X^\circ$  by  $\mathcal{J}_X^c$  and  $\mathcal{J}_X^\circ$ , respectively. Denote the pullback of  $\mathcal{P}_{g,n}$  to  $X^\circ$  by  $\mathcal{P}_X$ . Then  $\alpha$  defines an isomorphism of  $\mathcal{P}_X$  with  $\mathcal{J}_X^\circ$ . Under this isomorphism  $G_{\mathbf{d}}$  corresponds to the section

$$F_{\mathbf{d}} : [C; x_1, \dots, x_n] \mapsto -\alpha + \sum_{j=1}^n d_j x_j \in \text{Jac } C$$

and the pullback of the divisor  $W$  to  $\mathcal{P}_X$  corresponds to the divisor  $\Theta_\alpha$  which is defined locally by the theta function  $\vartheta_\alpha(z, \Omega)$  that corresponds to  $\alpha$ .

The section  $F_{\mathbf{d}}$  extends to a section of  $\mathcal{J}_X$  over  $X$  and the divisor  $\Theta_\alpha$  extends to  $\mathcal{J}_X$ . One therefore has the class  $F_{\mathbf{d}}^* \Theta_\alpha$  in  $H^2(X)$ . Its restriction to  $H^2(X^\circ)$  is the pullback of the class of  $W_{\mathbf{d}}$ .

**Theorem 11.7.** *The class of  $F_{\mathbf{d}}^* \Theta_\alpha$  in  $H^2(X)$  is*

$$\begin{aligned} \delta_0/8 - \lambda_1 + \sum_{j=1}^n (d_j + d_j^2) \psi_j/2 - \sum_{P \subseteq I} \sum_{\{x_j, x_k\} \subseteq P} d_j d_k \delta_0^P \\ - \frac{1}{4} \sum_{P \subseteq I} \sum_{h=1}^{g-1} (d_P^2 - (2h-1)d_P + h^2 - h) \delta_h^P. \end{aligned}$$

The class  $-\delta_0/8 + F_{\mathbf{d}}^* \Theta_\alpha$  is the pullback of an integral class from  $\overline{\mathcal{M}}_{g,n}$  as  $\delta_h^P = \delta_{g-h}^{P^c}$  and because each of  $d_j + d_j^2$ ,  $d_P^2 - (2h-1)d_P$  and  $h^2 - h$  is even.

*Proof.* The first step is to show that the class  $\theta_\alpha$  of  $\Theta_\alpha$  satisfies

$$(9) \quad \theta_\alpha = \hat{\phi}_H + \lambda_1/2 \in H^2(\mathcal{J}_g'),$$

where  $\mathcal{J}_g'$  denotes the universal jacobian over  $\overline{\mathcal{M}}_g - \Delta_0^{\text{sing}}$ . Granted this, the result follows from Theorem 11.6 as

$$F_{\mathbf{d}}^*(\theta_\alpha) = F_{\mathbf{d}}^*(\hat{\phi}_H + \lambda_1/2) = \lambda_1/2 + F_{\mathbf{d}}^*(\hat{\phi}_H)$$

and

$$\kappa_1 = 12\lambda_1 - \delta = 12\lambda_1 - \delta_o - \frac{1}{2} \sum_{j=1}^h \sum_{P \subseteq I} \delta_h^P.$$

To prove (9), first note that the relation  $\theta_\alpha = \phi_H + \lambda_1/2$  holds in  $H^2(\mathcal{J}_g^c)$ . This is because the restrictions of  $\theta_\alpha$  and  $\phi_H$  to each fiber of  $\mathcal{J}_g^c \rightarrow \mathcal{M}_g^c$  are equal and because the restriction of  $\phi_H$  to the zero section vanishes, while the restriction of  $\theta_\alpha$  to the zero section has class  $\lambda_1/2$  as the corresponding theta null  $\vartheta_\alpha(0, \Omega)$  is a modular form of weight  $1/2$  for some finite index subgroup of  $\text{Sp}_g(\mathbb{Z})$ . Since  $\hat{\phi}_H$ ,  $\theta_\alpha$  and  $\lambda_1$  are all classes of line bundles, and since  $\mathcal{J}_g - \mathcal{J}_g^c$  is the restriction of  $\mathcal{J}_g'$  to  $\Delta_0$ , it follows that

$$\theta_\alpha = \hat{\phi}_H + \lambda_1/2 + c\delta_0 \in H^2(\mathcal{J}_g')$$

Restricting both sides to the zero section implies that

$$\lambda_1/2 = 0 + \lambda_1/2 + c\delta_0 \in H^2(\overline{\mathcal{M}}_g - \Delta_0^{\text{sing}}) \cong H^2(\overline{\mathcal{M}}_g),$$

which implies that  $c = 0$ , as required.  $\square$

## 12. ELIASHBERG'S PROBLEM IN GENUS 1

The solution of Eliashberg's problem over  $\mathcal{M}_{g,n}^c$  in genus  $\geq 2$  given in the previous section fails when  $g = 1$  as we cannot divide by  $2g - 2$ . However, a variant of the methods of the previous section gives a complete solution in genus 1.

When  $g = 1$ , the class of  $F_{\mathbf{d}}^* \eta_1$  naturally lives in  $H^2(\overline{\mathcal{M}}_{1,n+1})$  as the locus of indeterminacy  $\Delta_0^{\text{sing}}$  of  $F_{\mathbf{d}}$  has codimension  $\geq 2$ , whereas  $F_{\mathbf{d}} \eta_1$  is the class of a divisor, and thus extends uniquely from  $\overline{\mathcal{M}}_{1,n+1} - \Delta_0^{\text{sing}}$  to  $\overline{\mathcal{M}}_{1,n+1}$ .

**Theorem 12.1.** *If  $\mathbf{d} = (d_0, \dots, d_n) \in \mathbb{Z}^{n+1}$  satisfies  $\sum_j d_j = 0$ , then*

$$F_{\mathbf{d}}^* \eta_1 = (-1 + (d_0^2 + \dots + d_n^2)/2) \lambda_1 - \sum_{P \subseteq I} \sum_{\{x_j, x_k\} \subseteq P} d_j d_k \delta_0^P \in H^2(\overline{\mathcal{M}}_{1,n+1}).$$

The restriction of this class to  $\mathcal{M}_{1,n+1}$  has been independently computed by Cavalieri and Marcus [5] using different methods.

Denote the universal elliptic curve  $\mathcal{J}_1 \rightarrow \overline{\mathcal{M}}_{1,1}$  by  $\mathcal{E}$ . Note that  $\mathcal{E} = \overline{\mathcal{M}}_{1,2} - \Delta_0^{\text{sing}}$ . Its restriction to  $\mathcal{M}_{1,1}$  is the universal elliptic curve  $\mathcal{E}^c$  of “compact type”. Since  $\mathcal{E}$  and  $\overline{\mathcal{M}}_{1,2}$  differ in codimension 2, their second cohomology and Picard groups are isomorphic:

$$H^2(\overline{\mathcal{M}}_{1,2}) \cong (\text{Pic } \overline{\mathcal{M}}_{1,2}) \otimes \mathbb{Q} \cong (\text{Pic } \mathcal{E}) \otimes \mathbb{Q} \cong H^2(\mathcal{E}).$$

These groups are 2-dimensional with basis  $\delta := \delta_0^{\{x_1, x_2\}}$ , the class of the zero section  $D$  of  $\mathcal{E} \rightarrow \overline{\mathcal{M}}_{1,1}$ , and  $\delta_0$ , the class of the fiber over the cusp of  $\overline{\mathcal{M}}_{1,1}$ . The class  $\lambda_1$  of the Hodge bundle is  $\delta_0/12$ .

We will deduce Theorem 12.1 from the corresponding result for the  $n$ th power

$$\mathcal{E}^n = \underbrace{\mathcal{E} \times_{\overline{\mathcal{M}}_{1,1}} \cdots \times_{\overline{\mathcal{M}}_{1,1}} \mathcal{E}}_n.$$

of the universal elliptic curve over  $\overline{\mathcal{M}}_{1,1}$ . A point in  $\mathcal{E}^n$  corresponds to the isomorphism class of an  $n$ -pointed elliptic curve  $(E; x_0, x_1, \dots, x_n)$  where  $x_0 = 0$ , the identity.<sup>9</sup> Let  $p : \mathcal{E}^n \rightarrow \overline{\mathcal{M}}_{1,1}$  be the canonical projection. For each  $\mathbf{d} = (d_0, \dots, d_n)$  as above, there is a section of the pullback  $p^* \mathcal{E} \rightarrow \mathcal{E}^n$  of the universal elliptic curve defined by

$$F_{\mathbf{d}}(E; x_0, \dots, x_n) = \sum_{j=0}^n d_j x_j = \sum_{j=1}^n d_j x_j \in E.$$

For  $0 \leq j < k \leq n$  let  $D_{j,k}$  be the divisor in  $\mathcal{E}^n$  where  $x_j = x_k$ . Denote its class in  $H^2(\mathcal{E}^n)$  by  $\delta_{j,k}$ . Let  $\Delta_0$  be the inverse image of the moduli point of the nodal cubic under the projection  $\mathcal{E}^n \rightarrow \overline{\mathcal{M}}_{1,1}$ . Then

$$H^2(\mathcal{E}^n) \cong (\text{Pic } \mathcal{E}^n) \otimes \mathbb{Q} = \mathbb{Q} \delta_0 \oplus \bigoplus_{0 \leq j < k \leq n} \mathbb{Q} \delta_{j,k}.$$

The pullback of  $\delta_{j,k}$  under the morphism  $\pi : \overline{\mathcal{M}}_{1,n+1} - \Delta_0^{\text{sing}} \rightarrow \mathcal{E}^n$  is

$$\pi^* \delta_{jk} = \sum_{\{x_j, x_k\} \subseteq P} \delta_0^P \in \text{Pic } \overline{\mathcal{M}}_{1,n+1}.$$

<sup>9</sup>Note that an  $n$ -pointed elliptic curves is an  $(n+1)$ -pointed genus 1 curve.



Since  $d_0 + \dots + d_n = 0$ ,  $d_0^2 + \dots + d_n^2 = -2 \sum_{0 \leq j < k \leq n} d_j d_k$ . Since  $F_{\mathbf{d}}^* \lambda_1 = \lambda_1$ , to prove Theorem 12.1 it suffices to prove that

$$(10) \quad F_{\mathbf{d}}^*(\delta + \lambda_1) = - \sum_{0 \leq j < k \leq n} d_j d_k (\delta_{j,k} + \lambda_1) \in H^2(\mathcal{E}^n).$$

The key step in the proof of this statement is to show that  $\delta + \lambda_1$  is a parallel, translation invariant class. This statement is made precise in the following lemma. For a positive integer  $e$ , let  $[e] : \mathcal{E} \rightarrow \mathcal{E}$  be multiplication by  $e$ .

**Lemma 12.2.** *The class  $\phi_H \in H^2(\mathcal{E}^c)$  extends uniquely to a class  $\hat{\phi}_H$  in  $H^2(\mathcal{E})$  that vanishes on the zero-section  $D$ . It is given by*

$$\hat{\phi}_H = \delta + \lambda_1 \in H^2(\mathcal{E}).$$

and is characterized by the two properties

$$\int_{\Delta_0} \hat{\phi}_H = 1 \text{ and } [e]^* \hat{\phi}_H = e^2 \hat{\phi}_H \text{ for all integers } e > 1.$$

*Proof.* The exact sequence

$$0 \rightarrow \mathbb{Q}\delta_0 \rightarrow H^2(\mathcal{E}) \rightarrow H^2(\mathcal{E}^c) \rightarrow 0.$$

is invariant under  $[e]^*$ , which acts trivially on the kernel and by multiplication by  $e^2$  on the quotient. Since  $\phi_H$  spans the right-hand group, it follows that it has a unique lift  $\hat{\phi}_H$  to  $H^2(\mathcal{E})$  with the property that  $[e]^* \hat{\phi}_H = e^2 \hat{\phi}_H$ .

Since  $[e]_*[D] = [D]$ , we have

$$(e^2 - 1) \int_D \hat{\phi}_H = \int_D [e]^* \hat{\phi}_H - \int_D \hat{\phi}_H = \int_{([e]^* - 1)D} \hat{\phi}_H = 0.$$

Since  $e > 1$ , this implies the vanishing  $\int_D \hat{\phi}_H = 0$  of  $\hat{\phi}_H$  on  $D$ . Since  $\Delta_0$  is the class of the fiber of  $\mathcal{E} \rightarrow \overline{\mathcal{M}}_{1,1}$ ,  $\int_{\Delta_0} \hat{\phi}_H = 1$ . Since  $\delta$  and  $\delta_0$  span  $H^2(\mathcal{E})$  and since its intersection pairing is non-singular, these two properties characterize  $\hat{\phi}_H$ .

To prove that  $\hat{\phi}_H = \delta + \lambda_1$ , it suffices to show that  $(\delta + \lambda_1) \cdot \delta = 0$  and that  $(\delta + \lambda_1) \cdot \delta_0 = 1$ . Since the Chen class of the normal bundle of  $D$  in  $\mathcal{E}$  is  $-\lambda_1$ , we have

$$\int_D \delta = D^2 = - \int_D \lambda_1,$$

which implies that  $(\delta + \lambda_1) \cdot \delta = 0$ . Since  $\delta_0$  is the class of a fiber of  $\mathcal{E} \rightarrow \overline{\mathcal{M}}_{1,1}$ ,  $\delta_0 \cdot \lambda_1 = 0$ , so that  $\delta_0 \cdot (\delta + \lambda_1) = \delta_0 \cdot \delta = 1$ .  $\square$

Each unordered pair  $\{j, k\}$  of integers in  $[0, n]$  determines a parallel class  $\hat{\phi}_{j,k} \in H^2(\mathcal{E}^n)$  as follows. Let  $p_{j,k} : \mathcal{E}^n \rightarrow \mathcal{E}$  be the projection that takes  $[E : x_0, \dots, x_n]$  to  $[E : x_j, x_k] = [E : 0, x_k - x_j]$ . Observe that  $D_{j,k} = p_{j,k}^* D$ . Set  $\hat{\phi}_{j,k} = p_{j,k}^* \hat{\phi}_H$ . Lemma 12.2 implies that  $\hat{\phi}_{j,k} = \delta_{j,k} + \lambda_1$ . Combining this with (10), we deduce that, to prove Theorem 12.1, it suffices to show that

$$(11) \quad F_{\mathbf{d}}^* \hat{\phi}_H = - \sum_{0 \leq j < k \leq n} d_j d_k \hat{\phi}_{j,k}.$$

**Lemma 12.3.** *The class  $\phi_{\Delta} \in H^2(\mathcal{E}^c \times_{\mathcal{M}_{1,1}} \mathcal{E}^c)$  extends to a class  $\hat{\phi}_{\Delta}$  in  $H^2(\mathcal{E}^2)$  with the property that  $[e]^* \hat{\phi}_{\Delta} = e^2 \hat{\phi}_{\Delta}$ . Specifically,*

$$(12) \quad \hat{\phi}_{\Delta} = p_{0,1}^* \hat{\phi}_H + p_{0,2}^* \hat{\phi}_H - p_{1,2}^* \hat{\phi}_H \in H^2(\mathcal{E}^2).$$

*Proof.* The class given by (12) is clearly an eigenvector of  $[e]$  with eigenvalue  $e^2$ . To prove the result, it suffices to prove that its restriction to  $(\mathcal{E}^c)^2 := \mathcal{E}^c \times_{\mathcal{M}_{1,1}} \mathcal{E}^c$  is  $\phi_\Delta$ . Since the restriction to  $(\mathcal{E}^c)^2$  of all classes in the formula are represented by parallel, translation invariant forms, it suffices to check the formula for a single smooth elliptic curve  $E$ . Let  $a, b$  be a symplectic basis of  $H^1(E)$ . Identify  $H^\bullet(E^2)$  with  $H^\bullet(E) \otimes H^\bullet(E)$ . Then a routine computation shows that the restriction of  $p_{1,1}^* \phi_H$  to  $E^2$  is

$$(a \wedge b) \otimes 1 + 1 \otimes (a \wedge b) - (a \otimes b - b \otimes a) = (p_{0,1}^* \phi_H + p_{0,2}^* \phi_H - \phi_\Delta)|_{E^2},$$

as required.  $\square$

For each  $0 \leq j < k \leq n$  define  $\hat{\phi}_\Delta^{j,k} = p_{j,k}^* \hat{\phi}_\Delta$ .

**Corollary 12.4.** *If  $0 < j < k \leq n$ , then  $\hat{\phi}_\Delta^{j,k} = \hat{\phi}_{0,j} + \hat{\phi}_{0,k} - \hat{\phi}_{j,k}$ .*

To prove (11), factor  $F_d : \mathcal{E}^n \rightarrow \mathcal{E}$  as follows:

$$\mathcal{E}^n \xrightarrow[(d_1, \dots, d_n)]{\quad F_d \quad} \mathcal{E}^n \xrightarrow{\text{tr}_n} \mathcal{E}$$

where  $(d_1, \dots, d_n) : [E; 0, x_1, \dots, x_n] \mapsto [E : 0, d_1 x_1, \dots, d_n x_n]$ . Note that the formula

$$\text{tr}_n^* \hat{\phi}_H = \sum_{j=1}^n \hat{\phi}_{0,j} + \sum_{1 \leq j < k \leq n} \hat{\phi}_\Delta^{j,k}$$

holds in  $H^2(\mathcal{E}^n)$  as it holds when restricted to  $(\mathcal{E}^c)^n$ , because both sides are eigenvectors of  $[e]$  with eigenvalue  $e^2$ , and because both sides vanish on the divisor  $x_1 + \dots + x_n = 0$  by an argument similar to the one used in the proof of Lemma 12.2.

Adapting the arguments of the previous section to this case, we see that

$$\begin{aligned} F_d^* \hat{\phi}_H &= \sum_{j=1}^n d_j^2 \hat{\phi}_{0,j} + \sum_{1 \leq j < k \leq n} d_j d_k \hat{\phi}_\Delta^{j,k} \\ &= \sum_{j=1}^n d_j^2 \hat{\phi}_{0,j} + \sum_{1 \leq j < k \leq n} d_j d_k (\hat{\phi}_{0,j} + \hat{\phi}_{0,k} - \hat{\phi}_{j,k}) \\ &= \sum_{j=1}^n (d_j^2 + (d_1 + \dots + d_n - d_j) d_j) \hat{\phi}_{0,j} - \sum_{1 \leq j < k \leq n} d_j d_k \hat{\phi}_{j,k} \\ &= \sum_{j=1}^n -d_0 d_j \hat{\phi}_{0,j} - \sum_{1 \leq j < k \leq n} d_j d_k \hat{\phi}_{j,k} \\ &= - \sum_{0 \leq j < k \leq n} d_j d_k \hat{\phi}_{j,k}. \end{aligned}$$

### 13. NORMAL FUNCTIONS AND POSITIVITY

In this section we show that if a variation of Hodge structure of weight  $-1$  is weakly polarized by  $S$ , then the pullback  $\nu^* \omega_S$  along a normal function  $\nu$  of the 2-form  $\omega_S$  on the total space that is naturally associated to the polarization is pointwise non-negative.

Suppose that  $\mathbb{V}$  is a polarized variation of Hodge structure over  $X$  of weight  $-1$  endowed with a weak polarization  $S$ .

**Theorem 13.1.** *If  $\nu : X \rightarrow J(\mathbb{V})$  is a normal function, then  $\nu^*\omega_S$  is a non-negative  $(1, 1)$ -form on  $X$ .*

*Proof.* As previously remarked,  $J(\mathbb{V})$  is isomorphic, as a bundle of tori, to  $\mathbb{V}_{\mathbb{R}}/\mathbb{V}_{\mathbb{Z}}$ . The normal function  $\nu$  thus corresponds to a section  $s : X \rightarrow \mathbb{V}_{\mathbb{R}}/\mathbb{V}_{\mathbb{Z}}$ . Locally this lifts to a section (also denoted by  $s$ ) of  $\mathbb{V}_{\mathbb{R}}$ . We can view this as a section of

$$\mathcal{V} := \mathbb{V} \otimes \mathcal{O}_X$$

This is a flat holomorphic vector bundle. Denote its Hodge filtration by

$$\mathcal{V} \supseteq \dots \supseteq \mathcal{F}^p \supseteq \mathcal{F}^{p+1} \supseteq \dots$$

Each  $\mathcal{F}^p$  is a holomorphic sub-bundle of  $\mathcal{V}$ . Since  $\mathbb{V}$  has weight  $-1$ , it splits as the sum

$$\mathcal{V} = \mathcal{F}^0 \oplus \overline{\mathcal{F}}^0,$$

where  $\overline{\mathcal{F}}^p$  denotes the complex conjugate of  $\mathcal{F}^p$  in  $\mathcal{V}$ . Note that  $\mathcal{F}^0$  is a holomorphic sub-bundle, while  $\overline{\mathcal{F}}^0$  is not, in general. Decompose  $s$  as

$$s = p + n$$

where  $p$  is a smooth section of  $\mathcal{F}^0$  and  $n$  is a smooth section of  $\overline{\mathcal{F}}^0$ . Since  $s$  is real,  $p$  and  $n$  are complex conjugates of each other.

We can compute the differentials of  $s$ ,  $n$  and  $p$  with respect to the flat structure on  $\mathbb{V}_{\mathbb{R}}$ . Since  $\omega_S$  is parallel, we have

$$\nu^*\omega_S = s^*\omega_S = S(ds, ds) = S(\partial s + \bar{\partial} s, \partial s + \bar{\partial} s) = 2S(\partial s, \bar{\partial} s).$$

This is clearly of type  $(1, 1)$ .

Next we prove that  $\nu^*\omega_S$  is non-negative. Since a 2-form is positive if and only if it is positive on every holomorphic arc in  $X$ , we may assume that  $X$  is the unit disk. Let  $t$  be a holomorphic coordinate in  $X$ .

Griffiths' infinitesimal period relation for normal functions implies that for any smooth local lift  $f : X \rightarrow \mathcal{V}$  of the normal function  $\partial f / \partial t \in \mathcal{F}^{-1}$ . Here, and in what follows, the partial derivatives are taken with respect to the natural flat connection on  $\mathcal{V}$ . Since  $n$  and  $p + n$  are both smooth local lifts of  $\nu$ ,

$$(13) \quad \frac{\partial p}{\partial t} \in \mathcal{F}^{-1} \text{ and } \frac{\partial n}{\partial t} \in \mathcal{F}^{-1}.$$

Since  $\mathcal{F}^0$  is a holomorphic sub-bundle of  $\mathcal{V}$ ,  $\partial p / \partial \bar{t} \in \mathcal{F}^0$ . Since  $n$  is the conjugate of  $p$ , we have  $\partial n / \partial t \in \overline{\mathcal{F}}^0$ . Combining this with Griffiths infinitesimal period relation (13), we conclude that  $\partial n / \partial t \in \mathcal{F}^{-1} \cap \overline{\mathcal{F}}^0$  and, by taking complex conjugates, that  $\partial p / \partial \bar{t} \in \mathcal{F}^0 \cap \overline{\mathcal{F}}^{-1}$ .

Next compute the pullback of  $\omega_S$ : as above,

$$\begin{aligned} \nu^*\omega_S &= 2S(\partial s, \bar{\partial} s) \\ &= 2S(\partial n + \partial p, \bar{\partial} n + \bar{\partial} p) \\ &= 2S\left(\frac{\partial n}{\partial t} + \frac{\partial p}{\partial t}, \frac{\partial n}{\partial \bar{t}} + \frac{\partial p}{\partial \bar{t}}\right) dt \wedge d\bar{t} \\ &= 2S(v(t), \overline{v(t)}) dt \wedge d\bar{t} \end{aligned}$$

where  $v(t) := \partial n / \partial t + \partial p / \partial t$ . Set  $t = x + iy$ . Since  $v(t) \in H_t^{-1,0}$  for all  $t$ , it follows that  $\nu^* \omega_S = 2i^{-1-0} S(v, \bar{v}) dx \wedge dy$ , which is non-negative.  $\square$

**Corollary 13.2.** *Suppose that  $\mathbb{V}$  is a variation of Hodge structure of weight  $-1$  over a smooth complex algebraic variety  $X$ . If  $S$  is a weak polarization of  $V$  and  $\nu$  is a normal function section of  $J(\mathbb{V}) \rightarrow X$ , then for all complete curves  $T$  in  $X$*

$$\int_T \nu^* \omega_S \geq 0$$

*with equality if and only if the infinitesimal invariant of the normal function vanishes.*

Suppose that  $\overline{X}$  is a smooth completion of  $X$ . If Conjecture 6.4 holds, then we can conclude that the natural extension of  $\nu^* \phi_S$  to a class in  $H^2(\overline{X})$  has non-negative degree on all complete curves  $T$  in  $\overline{X}$  that do not lie in  $\overline{X} - X$ .

Since  $F_{\mathbf{d}}$  is a normal function, the class  $F_{\mathbf{d}}^* \phi_H$  is represented by a non-negative  $(1, 1)$ -form on  $\mathcal{M}_{g,n}^c$ . Theorem 11.5 thus implies the following positivity statement:

**Proposition 13.3.** *If  $g \geq 2$  and  $(d_1, \dots, d_n) \in \mathbb{Z}^n$  satisfies  $\sum_{j=1}^n d_j = (2g - 2)m$ , then for each  $k \geq 1$ ,*

$$\left( -m^2 \kappa_1 / 2 + \sum_{j=1}^n (d_j m + d_j^2 / 2) \psi_j - \sum_{P \subseteq I} \sum_{\{x_j, x_k\} \subseteq P} d_j d_k \delta_0^P - \frac{1}{4} \sum_{P \subseteq I} \sum_{h=1}^{g-1} (d_P - (2h-1)m)^2 \delta_h^P \right)^k$$

*has non-negative degree on all complete  $k$ -dimensional subvarieties of  $\mathcal{M}_{g,n}^c$ .*

When  $k = g$  and  $m = g$ , this implies that the pullback  $F_{\mathbf{d}}^* \eta_g$  of the zero section has non-negative degree on all complete,  $g$ -dimensional subvarieties of  $\mathcal{M}_{g,n}^c$ .

#### 14. SLOPE INEQUALITIES

As an immediate consequence of Corollary 13.2 with the computations in Section 10, we obtain the following versions Moriwaki's inequalities [32, 33]. The second assertion below was obtained independently by the author in the late 1990s (cf. [33, p. 195]).

**Theorem 14.1.** *If  $g \geq 2$ , then*

(i) *the divisor*

$$(8g + 4)\lambda_1 - 4 \sum_{h=1}^{[g/2]} h(g-h)\delta_h$$

*has non-negative degree on each complete curve in  $\mathcal{M}_g^c$ ,*

(ii) *the divisors*

$$4g(g-1)\psi - \kappa_1 - \sum_{h=1}^{g-1} (2h-1)^2 \delta_{g-h}^{\{x\}}$$

$$8\lambda_1 + 4g\psi - \delta_0 - \sum_{h=1}^{g-1} h\delta_{g-h}^{\{x\}} \in H^2(\mathcal{M}_{g,1}^c)$$

*have non-negative degree on each complete curve in  $\mathcal{M}_{g,1}^c$ .*

The semi-positivity of the 2-forms representing these classes implies that their powers are also semi-positive.

**Corollary 14.2.** *If  $g \geq 2$  and  $k \geq 1$ , then*

(i) *the cohomology class*

$$\left( (8g+4)\lambda_1 - 4 \sum_{h=1}^{\lfloor g/2 \rfloor} h(g-h)\delta_h \right)^k \in H^{2k}(\mathcal{M}_g^c)$$

*has non-negative degree on each complete  $k$ -dimensional subvariety of  $\mathcal{M}_g^c$ .*

(ii) *the cohomology classes*

$$\left( 4g(g-1)\psi - \kappa_1 - \sum_{h=1}^{g-1} (2h-1)^2 \delta_h^{\{x\}} \right)^k$$

*and*

$$\left( 8\lambda_1 + 4g\psi - \delta_0 - \sum_{h=1}^{g-1} h\delta_h^{\{x\}} \right)^k$$

*in  $H^{2k}(\mathcal{M}_{g,1}^c)$  have non-negative degree on each complete  $k$ -dimensional subvariety of  $\mathcal{M}_{g,1}^c$ .*

This result also follows from Kleiman's criterion [27, Thm. 1], as János Kollár pointed out to me.

The statements in Theorem 14.1 are weaker than Moriwaki's in the sense that they apply only to complete curves in  $\mathcal{M}_{g,n}^c$  where  $n = 0, 1$ , but stronger than Moriwaki's as his versions apply only to complete curves in  $\overline{\mathcal{M}}_{g,n}$  that do not lie in the boundary divisor  $\Delta$ . These two versions suggest the following stronger version of Moriwaki's inequalities, which would follow from Conjecture 6.4 if it were true.

**Conjecture 14.3.** For all  $g \geq 2$ :

(i) the divisor

$$M := (8g+4)\lambda_1 - 4 \sum_{h=1}^{\lfloor g/2 \rfloor} h(g-h)\delta_h$$

*has non-negative degree on each complete curve in  $\overline{\mathcal{M}}_g$  that does not lie in  $\Delta_0$ ;*

(ii) the divisors

$$W_H := 4g(g-1)\psi - \kappa_1 - \sum_{h=1}^{g-1} (2h-1)^2 \delta_{g-h}^{\{x\}}$$

$$W_L := 8\lambda_1 + 4g\psi - \delta_0 - \sum_{h=1}^{g-1} h\delta_{g-h}^{\{x\}} \in H^2(\mathcal{M}_{g,1}^c)$$

*have non-negative degree on each complete curve in  $\overline{\mathcal{M}}_{g,1}$  that does not lie in  $\Delta_0$ ;*

(iii) and for all  $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{Z}^d$  with  $\sum d_j = m$ , the class

$$-m^2 \kappa_1 / 2 + \sum_{j=1}^n (d_j m + d_j^2 / 2) \psi_j - \sum_{P \subseteq I} \sum_{\{x_j, x_k\} \subseteq P} d_j d_k \delta_0^P - \frac{1}{4} \sum_{P \subseteq I} \sum_{h=1}^{g-1} (d_P - (2h-1)m)^2 \delta_h^P$$

has non-negative degree on all complete curves in  $\overline{\mathcal{M}}_{g,n}$  that do not lie in  $\Delta_0$ .

**14.1. The jumping divisor.** To better understand the behaviour of the degree of the Moriwaki divisors on curves in  $\overline{\mathcal{M}}_{g,n}$  that pass through  $\Delta_0^{\text{sing}}$  but are not contained in  $\Delta_0$ , we need to consider the phenomenon of “height jumping” and the associated notion of a *jumping divisor*.

Suppose that  $X = \overline{X} - D$  where  $\overline{X}$  is a smooth projective variety and  $D$  is a normal crossings divisor in  $\overline{X}$ . Suppose that  $\mathbb{U}$  is a weight  $-1$  variation of Hodge structure over  $X$  that is polarized by  $S$ . Suppose that  $\nu$  is a normal function section of  $J(\mathbb{U})$ . Then one has the symmetric biextension line bundle  $\nu^* \widehat{\mathcal{B}}$  over  $X$ . Lear’s Theorem implies that a positive power  $\widehat{\mathcal{B}}_{N,\nu}$  of  $\nu^* \widehat{\mathcal{B}}$  extends naturally to a line bundle over  $\overline{X}$ . It is characterized by the property that the metric on  $\nu^* \widehat{\mathcal{B}}$  extends to a continuous metric on this line bundle over  $\overline{X} - D^{\text{sing}}$ . For clarity of exposition, we suppose that the power is 1, so that  $\nu^* \widehat{\mathcal{B}}$  itself extends.<sup>10</sup> Denote the extended line bundle  $\widehat{\mathcal{B}}_{1,\nu}$  by  $\mathcal{B}_{\overline{X}}$ .

Now suppose that  $f : T \rightarrow \overline{X}$  is a morphism from a smooth projective curve to  $\overline{X}$  whose image is not contained in  $D$ . Set  $T' = T - f^{-1}(D)$ . Applying Lear’s Theorem to the normal function section  $f^* \nu$  of  $J(f^* \mathbb{U}) \rightarrow T'$ , we obtain the Lear extension  $\mathcal{B}_T$  of  $(\nu \circ f)^* \widehat{\mathcal{B}}$  to  $T$ . If the image of  $T$  avoids  $D^{\text{sing}}$ , then  $f^* \mathcal{B}_{\overline{X}} \cong \mathcal{B}_T$ . This is because  $\mathcal{B}_{\overline{X}}$  is metrized over  $\overline{X} - D^{\text{sing}}$ , so that its pullback to  $T$  is the unique metrized extension to  $T$  of  $(\nu \circ f)^* \widehat{\mathcal{B}}$ . In general, there is a 0-cycle  $J$  on  $T$ , supported on  $T - T'$ , such that

$$f^* \mathcal{B}_{\overline{X}} \cong \mathcal{B}_T(J).$$

If  $f(T) \cap D^{\text{sing}}$  is empty, then  $J = 0$ . We will call  $J$  the *jumping divisor* of  $\nu$  on  $T$ . It is not always trivial, as we shall explain below. The jumping divisor encodes “height jumping”.

**14.2. Height jumping.** Set  $d = \dim X$ . Assume that  $\mathbb{D}^d$  is a polydisk in  $\overline{X}$  with coordinates  $(t_1, \dots, t_d)$ , and that its intersection with  $D$  is the divisor  $t_1 \dots t_m = 0$ . Assume that the monodromy of  $\mathbb{U}$  about the branch  $t_j = 0$  of  $D$  is unipotent for each  $j \in \{1, \dots, m\}$ .<sup>11</sup> Suppose that

$$\beta \in H^0((\mathbb{D}^*)^m \times \mathbb{D}^{d-m}, \nu^* \widehat{\mathcal{B}})$$

is a biextension section of  $\nu^* \widehat{\mathcal{B}}$  defined over the punctured polydisk. The associated *height function*  $(\mathbb{D}^*)^m \times \mathbb{D}^{d-m} \rightarrow \mathbb{R}^+$  is the function

$$(t_1, \dots, t_d) \mapsto \log |\beta(t_1, \dots, t_d)|.$$

<sup>10</sup>Otherwise, replace it by the power that does extend. In all known examples the power is 1.

<sup>11</sup>This condition is satisfied by the variations  $\mathbb{H}$ ,  $\mathbb{L}$  and  $\mathbb{V}$  over  $\mathcal{M}_{g,n}$ .

Suppose that  $P \in T$  and that  $f(P)$  is the origin of  $\mathbb{D}^d$ . Suppose that  $\mathbb{D}$  is a disk in  $T$  with coordinate  $t$  with  $t(P) = 0$ . The restriction of  $f$  to  $\mathbb{D}$  is a holomorphic arc  $f : \mathbb{D} \rightarrow \mathbb{D}^d$ . Set

$$r_j := \text{ord}_{t=0} f^* t_j, \quad j = 1, \dots, m.$$

There is a rational number  $q(r_1, \dots, r_m)$ , which depends only on the exponents  $r_j$ , such that

$$\log |\beta(f(t))|_{\mathcal{B}} \sim q(r_1, \dots, r_m) \log |t|.$$

One might expect that  $q(r_1, \dots, r_m)$  is linear. Surprisingly, this is not the case.

To better understand this, write

$$q(r_1, \dots, r_m) = q_0(r_1, \dots, r_m) + j(r_1, \dots, r_m)$$

where  $q_0$  is the linear function

$$q_0(r_1, \dots, r_m) = \sum_{j=1}^m r_j q(\mathbf{e}_j)$$

and  $\mathbf{e}_1, \dots, \mathbf{e}_m$  is the standard basis of  $\mathbb{Z}^m$ . We shall call  $j$  the *jump function* of  $\beta$  at  $P$ . When  $j$  vanishes, the height behaves as expected. Surprisingly, the height can jump. I first observed this when trying to understand Moriwaki's inequality, as explained in the following example. Although I was aware of height jumping through this example, I had no explanation for it. Recently Brosnan and Pearlstein [2] have given a complete explanation of this phenomenon.

For the curve  $f : T \rightarrow \overline{X}$  and  $P \in T$ , define  $j_P = j(r_1, \dots, r_m)$ . The jumping function determines the jumping divisor.

**Proposition 14.4.** *The jumping divisor  $J$  of  $\nu$  on  $T$  associated to  $f : T \rightarrow \overline{X}$  is the 0-cycle*

$$J = \sum_{P \in T} j_P P$$

on  $T$ .

This is proved, using techniques similar to those described in [23, §8], by considering the asymptotics as one approaches  $P$  of the length

$$|s|_{\overline{X}} / |s|_T$$

of a section  $s$  of  $f^* \mathcal{B}_{\overline{X}} \otimes \mathcal{B}_T^{-1}$  that trivializes it over  $T'$ . Here  $|\cdot|_{\overline{X}}$  denote the natural metric of  $(\nu \circ f)^* \widehat{\mathcal{B}}$  over  $T'$  and  $|\cdot|_T$  denotes the natural metric on  $\mathcal{B}_T$ .

**14.3. An example of height jumping.** In this example,  $g \geq 3$ ,  $X = \mathcal{M}_g^c$ ,  $\mathbb{U}$  is the variation  $\mathbb{V}$  and  $\nu$  is the normal function associated to the Ceresa cycle. Denote the Lear extension of  $\nu^* \widehat{\mathcal{B}}$  to  $\overline{\mathcal{M}}_g$  by  $\mathcal{B}_{\overline{\mathcal{M}}}$ . The main result of [23] implies that  $c_1(\mathcal{B}_{\overline{\mathcal{M}}})$  is the Moriwaki divisor

$$M := (8g + 4)\lambda_1 - g\delta_0 - \sum_{h=1}^{\lfloor g/2 \rfloor} 4h(g - h)\delta_h$$

Let  $\overline{\mathcal{H}}_g := \{[C] \in \overline{\mathcal{M}}_g : C \text{ is hyperelliptic}\}$  be the locus of hyperelliptic curves in  $\overline{\mathcal{M}}_g$ . Set  $\mathcal{H}_g^c = \overline{\mathcal{H}}_g \cap \mathcal{M}_g^c$ . The normal function  $\nu$  vanishes identically on  $\mathcal{H}_g^c$ . This implies that the line bundle  $\nu^* \widehat{\mathcal{B}}$  over  $\mathcal{H}_g^c$  is trivial as metrized holomorphic line bundle. Consequently, its Lear extension to  $\overline{\mathcal{H}}_g$ , which we denote by  $\mathcal{B}_{\overline{\mathcal{H}}}$ , is a trivial as a metrized holomorphic line bundle over  $\overline{\mathcal{H}}_g$ .

Recall that the boundary  $\overline{\mathcal{H}}_g - \mathcal{H}_g$  of  $\overline{\mathcal{H}}_g$  is a union of divisors

$$\Delta_h, \quad 1 \leq h \leq g/2 \text{ and } \Xi_h, \quad 0 \leq h \leq (g-1)/2,$$

where  $\Delta_h$  is the restriction of the boundary divisor  $\Delta_h$  of  $\overline{\mathcal{M}}_g$  to  $\overline{\mathcal{H}}_g$ ; where  $\Xi_0$  is the divisor whose generic point is an irreducible, geometrically connected hyperelliptic curve with one node; and where  $\Xi_h$ , ( $h \neq 0$ ) is the locus whose generic point is a hyperelliptic curve with two nodes that are conjugate under the hyperelliptic involution and whose normalization is the union of two smooth, geometrically connected hyperelliptic curves, one of genus  $h$  and the other of genus  $g-h-1$ . Denote the class of  $\Xi_h$  in  $H^2(\overline{\mathcal{H}}_g)$  by  $\xi_h$ .

Since  $\mathcal{B}_{\overline{\mathcal{X}}}$  is metrized over  $\overline{\mathcal{M}}_g - \Delta_0^{\text{sing}}$ , the restrictions of  $\mathcal{B}_{\overline{\mathcal{M}}}$  and  $\mathcal{B}_{\overline{\mathcal{H}}}$  to

$$\overline{\mathcal{H}}_g - \bigcup_{h>0} \Xi_h$$

are isomorphic as metrized line bundles. This implies that

$$j^* \mathcal{B}_{\overline{\mathcal{M}}} \otimes \mathcal{B}_{\overline{\mathcal{H}}}^{-1} \cong \mathcal{O}(J)$$

where  $j$  denotes the inclusion  $\overline{\mathcal{H}}_g \hookrightarrow \overline{\mathcal{M}}_g$  and where  $J$  is a linear combination of the  $\Xi_h$ ,  $h > 0$ . Note that, since  $\mathcal{B}_{\overline{\mathcal{H}}}$  is trivial,

$$\mathcal{O}(J) \cong j^* \mathcal{B}_{\overline{\mathcal{M}}} \cong \mathcal{O}(M)|_{\overline{\mathcal{H}}_g} \in \text{Pic } \overline{\mathcal{H}}_g.$$

The restriction of the Moriwick divisor to  $\overline{\mathcal{H}}_g$  is easily seen to be

$$j^* M = (8g+4)\lambda_1 - g\xi_0 - \sum_{h=1}^{\lfloor (g-1)/2 \rfloor} 2g\xi_h - 4 \sum_{h=1}^{\lfloor g/2 \rfloor} h(g-h)\delta_h.$$

On the other hand, Cornalba and Harris [6] have shown that

$$(8g+4)\lambda_1 - g\xi_0 - \sum_{h=1}^{\lfloor (g-1)/2 \rfloor} 2(h+1)(g-h)\xi_h - 4 \sum_{h=1}^{\lfloor g/2 \rfloor} h(g-h)\delta_h = 0$$

in  $\text{Pic } \overline{\mathcal{H}}_g$ . Together these imply that

$$J = \sum_{h=1}^{\lfloor (g-1)/2 \rfloor} 2h(g-h-1)\Xi_h.$$

It is now easy to construct an example of height jumping. Suppose that  $f : T \rightarrow \overline{\mathcal{M}}_g$  is a curve whose image lies in the hyperelliptic locus and is not contained in  $\Delta_0$ . If  $h > 0$  and if the image of  $f$  intersects  $\Xi_h$  transversely at a smooth point  $f(P)$ , then the computations above imply that

$$j_P = 2(h+1)(g-h) - 2g = 2h(g-h-1) > 0.$$

This implies that

$$\deg_T M = \deg_T J > 0.$$

Moriwaki's inequality and positivity in Hodge theory suggest that the jumping divisor associated to any curve in  $\overline{\mathcal{M}}_g$  should be effective. Denote by  $J_T$  the jumping divisor associated to a morphism  $f : T \rightarrow \overline{\mathcal{M}}_g$  whose image is not contained in  $\Delta_0$ .

**Conjecture 14.5** (weak form). For all projective curves  $f : T \rightarrow \overline{\mathcal{M}}$  whose image is not contained in  $\Delta_0$ , the jumping divisor  $J_T$  is effective.



This and Conjecture 6.4, if true, imply a stronger version of Moriwaki's inequalities as, for example,

$$\deg_T M = \deg_T \mathcal{B}_T + \deg_T J_T \geq \deg_T J_T \geq 0.$$

This would imply that the degree of Moriwaki's divisor on most curves not contained in  $\Delta_0$  that pass through  $\Delta_0^{\text{sing}}$  would be strictly positive as  $\deg_T M$  would be bounded below by the degree of its jumping divisor.

Similarly, one can conjecture that for all projective curves  $f : T \rightarrow \overline{\mathcal{M}}_{g,1}$  whose image is not contained in  $\Delta_0$ , the jumping divisor associated to the biextension line bundle associated to the normal function section  $\mathcal{K}$  of  $J(\mathbb{H})$  is always effective.

In general, one might hope that in the situation described in Paragraph 14.1, the jumping divisor  $J$  associated to a curve  $f : T \rightarrow \overline{X}$  whose image is not contained in  $D$ , is effective.

#### APPENDIX A. NORMAL FUNCTIONS OVER $\mathcal{M}_{g,n}$

For completeness, we state the classification (mod torsion) of normal functions over  $\mathcal{M}_{g,n}$  associated to variations of Hodge structure whose monodromy representation factors through a rational representation of  $\text{Sp}_g$ . It follows quite directly from results proved in [15, §8].<sup>12</sup>

The isomorphism classes of irreducible rational representations of the  $\mathbb{Q}$ -group  $\text{Sp}_g$  are indexed by partitions  $\lambda$

$$n = \lambda_1 + \cdots + \lambda_h, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_h, \quad h \leq g$$

of integers  $n$  into at most  $g$  parts. Denote the local system over  $\mathcal{A}_g$  that corresponds to the partition  $\lambda$  by  $\mathbb{V}_\lambda$ . It underlies a  $\mathbb{Q}$ -variation of Hodge structure of weight  $-|\lambda|$ , where

$$|\lambda| := \lambda_1 + \cdots + \lambda_h.$$

These can be pulled back to variations of Hodge structure over  $\mathcal{M}_{g,n}$  along the period map. Note that  $\mathbb{H} = \mathbb{V}_{[1]}$  and that  $\mathbb{V} = \mathbb{V}_{[1^3]}(-1)$ .

Recall that  $\Gamma A$  denotes the set of rational  $(0,0)$  classes of a  $\mathbb{Q}$ -Hodge structure  $A$ . If  $A$  is polarized, then  $A = \Gamma A \oplus (\Gamma A)^\perp$  in the category of  $\mathbb{Q}$ -Hodge structures.

**Theorem A.1.** *Suppose that  $2g - 2 + n > 0$ . If  $\mathbb{U}$  is a polarized variation of Hodge structure over  $\mathcal{M}_{g,n}$  whose monodromy representation factors through a rational representation of  $\text{Sp}_g$ , then there is an isomorphism of variations of  $\mathbb{Q}$ -Hodge structure*

$$\mathbb{U} \cong \bigoplus_{\lambda} A_{\lambda} \otimes_{\mathbb{Q}} \mathbb{V}_{\lambda},$$

where  $A_{\lambda}$  is the Hodge structure  $H^0(\mathcal{M}_{g,n}, \text{Hom}(\mathbb{V}_{\lambda}, \mathbb{U}))$ . Moreover, if  $A$  is a polarized Hodge structure, then

- (i)  $\text{Ext}_{\text{MHS}(\mathcal{M}_{g,n})}^1(\mathbb{Q}(0), A_{\lambda} \otimes \mathbb{V}_{\lambda})$  vanishes unless  $\lambda = [1]$  or  $\lambda = [1^3]$ ,
- (ii)  $\text{Ext}_{\text{MHS}(\mathcal{M}_{g,n})}^1(\mathbb{Q}(0), A \otimes \mathbb{H})$  vanishes when  $\Gamma A = 0$ .
- (iii)  $\text{Ext}_{\text{MHS}(\mathcal{M}_{g,n})}^1(\mathbb{Q}(0), \mathbb{H})$  has basis  $\mathcal{K}_1, \dots, \mathcal{K}_n$ ,
- (iv)  $\text{Ext}_{\text{MHS}(\mathcal{M}_{g,n})}^1(\mathbb{Q}(0), A \otimes \mathbb{V})$  vanishes when  $\Gamma A = 0$ ,
- (v)  $\text{Ext}_{\text{MHS}(\mathcal{M}_{g,n})}^1(\mathbb{Q}(0), \mathbb{V})$  is one-dimensional, spanned by  $\nu$ .

<sup>12</sup>Note that there are two typos on page 121. Line 4 should begin  $\dim \Gamma H^1(\Gamma_{g,r}^n, V(\lambda))$ , and there is a 2 missing from the right-hand side of line -7.

Crudely stated, this result says that all normal functions over  $\mathcal{M}_{g,n}$  associated to variations of Hodge structure that are representations of  $\mathrm{Sp}_g$  can be expressed, modulo torsion, as rational linear combinations of the basic normal functions  $\nu$  and  $\mathcal{K}_1, \dots, \mathcal{K}_n$ . For example,  $(2g-2)\delta_{j,k} = \mathcal{K}_j - \mathcal{K}_k$ .

## APPENDIX B. THE BIG PICTURE

The philosophy behind this work is that a significant amount of the geometry of  $\overline{\mathcal{M}}_{g,n}$  is encoded in the category  $\mathbf{MHS}_{g,n}$  of those admissible variations of  $\mathbb{Z}$ -MHS over  $\overline{\mathcal{M}}_{g,n}$  whose weight graded quotients are subquotients of Tate twists  $\mathbb{H}^{\otimes m}(d)$  of tensor powers of the fundamental local system

$$\mathbb{H} := R^1\pi_*\mathbb{Z}(1)$$

associated to the universal curve  $\pi : \mathcal{C} \rightarrow \mathcal{M}_{g,n}$ . The normal functions discussed in this paper are objects of  $\mathbf{MHS}_{g,n}$ .

The pure objects of  $\mathbf{MHS}_{g,n}$  are the variations of Hodge structure that correspond to the irreducible representations of the group  $\mathrm{GSp}_g$  of symplectic similitudes.<sup>13</sup> These are all pulled back from variations of Hodge structure over  $\mathcal{A}_g$  along the period mapping  $\mathcal{M}_{g,n} \rightarrow \mathcal{A}_g$ . Theorem A.1 implies that the only non-trivial extensions between these (mod torsion) are of the form

$$(14) \quad 0 \rightarrow \mathbb{V}_b \rightarrow \mathbb{E} \rightarrow \mathbb{V}_a \rightarrow 0$$

where the weights  $w_a$  and  $w_b$  of the pure variations  $\mathbb{V}_a$  and  $\mathbb{V}_b$  satisfy  $w_a = 1 + w_b$ . This is because the extension (14) is determined by the extension

$$0 \rightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{V}_a, \mathbb{V}_b) \rightarrow \mathbb{E}' \rightarrow \mathbb{Z}_{\mathcal{M}_{g,n}}(0) \rightarrow 0$$

obtained by applying  $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{V}_a, \cdot)$  to (14) and then pulling back along the identity  $\mathbb{Z}(0) \rightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{V}_a, \mathbb{V}_a)$ . That is, the 1-extensions in  $\mathbf{MHS}_{g,n}$  correspond to normal functions (mod torsion) over  $\mathcal{M}_{g,n}$ .

The question arises as to how one can understand  $\mathbf{MHS}_{g,n}$ . A first approximation is to understand the monodromy representations of objects of  $\mathbf{MHS}_{g,n}$ . The (orbifold) fundamental group of  $\mathcal{M}_{g,n}$  is the mapping class group:

$$\pi_1(\mathcal{M}_{g,n}, [C; P]) \cong \Gamma_{C,P} := \pi_0 \mathrm{Diff}^+(C, P),$$

where  $P = \{x_1, \dots, x_n\}$  and  $\pi_0 \mathrm{Diff}^+(C, P)$  denotes the group of connected components of the group of orientation preserving diffeomorphisms of  $C$  that fix  $P$  pointwise.<sup>14</sup> The action of  $\Gamma_{C,P}$  on  $H_1(C, \mathbb{Z})$  induces a homomorphism

$$\rho : \Gamma_{C,P} \rightarrow \mathrm{Aut}(H_1(C, \mathbb{Z}), \text{intersection form}) =: \mathrm{Sp}(H_1(C, \mathbb{Z})) \cong \mathrm{Sp}_g(\mathbb{Z})$$

which is well known to be surjective. Its kernel is the *Torelli group*,  $T_{C,P}$ .

Because each object  $\mathbb{V}$  of  $\mathbf{MHS}_{g,n}$  is filtered by its weight filtration

$$\cdots \subseteq W_{j-1}\mathbb{V} \subseteq W_j\mathbb{V} \subseteq W_{j+1}\mathbb{V} \subseteq \cdots$$

where each  $\mathrm{Gr}_j^W \mathbb{V} := W_j\mathbb{V}/W_{j-1}\mathbb{V}$  is a pure variation of Hodge structure, the Zariski closure (over  $\mathbb{Q}$ ) of its monodromy representation

$$\rho_V : \Gamma_{C,P} \rightarrow \mathrm{Aut}(V_{C,P})$$

<sup>13</sup>This is well known. An explanation can be found in [19, §8.1].

<sup>14</sup>This group depends only on  $g$  and  $n$  and is often denoted by  $\Gamma_{g,n}$ .

is an extension

$$1 \rightarrow U_V \rightarrow G_V \rightarrow \mathrm{Sp}(H_1(C)) \rightarrow 1$$

of algebraic  $\mathbb{Q}$ -groups where  $U_V$  is unipotent. It is thus natural to consider all Zariski dense representations  $\Gamma_{C,P} \rightarrow G(\mathbb{Q})$  where  $G$  is a  $\mathbb{Q}$  algebraic group that is an extension of  $\mathrm{Sp}(H_1(C))$  by a unipotent group and where  $\Gamma_{C,P} \rightarrow G \rightarrow \mathrm{Sp}(H_1(C))$  is the standard representation  $\rho$ . These form an inverse system. Their inverse limit is known as the *completion of  $\Gamma_{C,P}$  relative to  $\rho$* ; it is studied in [16].

The relative completion of  $\Gamma_{C,P}$  with respect to  $\rho$  is an extension

$$1 \rightarrow \mathcal{U}_{C,P} \rightarrow \mathcal{G}_{C,P} \rightarrow \mathrm{Sp}(H_1(C)) \rightarrow 1$$

of affine  $\mathbb{Q}$ -groups, where  $\mathcal{U}_{C,P}$  is prounipotent. There is a canonical homomorphism  $\Gamma_{C,P} \rightarrow \mathcal{G}_{C,P}(\mathbb{Q})$ . Denote the corresponding sequence of Lie algebras by

$$(15) \quad 0 \rightarrow \mathfrak{u}_{C,P} \rightarrow \mathfrak{g}_{C,P} \rightarrow \mathfrak{sp}(H_1(C)) \rightarrow 0.$$

It is proved in [17] and [16] that for each choice of a base point  $[C, P]$  of  $\mathcal{M}_{g,n}$ , the sequence (15) is a short exact sequence of Lie algebras in MHS.

Define a *Hodge representation* of  $\Gamma_{C,P}$  to be a MHS  $V$  and a pair of representations  $\phi_{\mathbb{Z}} : \Gamma_{C,P} \rightarrow \mathrm{Aut}_{\mathbb{Z}} V$  and  $\phi : \mathcal{G}_{C,P} \rightarrow \mathrm{Aut} V$  such that the diagram

$$\begin{array}{ccc} \Gamma_{C,P} & \xrightarrow{\phi_{\mathbb{Z}}} & \mathrm{Aut}_{\mathbb{Z}} V \\ \downarrow & & \downarrow \\ \mathcal{G}_{C,P}(\mathbb{Q}) & \xrightarrow{\phi} & \mathrm{Aut}_{\mathbb{Q}} V \end{array}$$

commutes and the induced homomorphism  $d\phi : \mathfrak{g}_{C,P} \rightarrow \mathrm{End} V$  is a morphism of MHS. The main result of [21] implies that  $\mathrm{MHS}_{g,n}$  is equivalent to the category of Hodge representations of  $\mathfrak{g}_{C,P}$ , a statement informally conjectured by Deligne. Because of this, extensions in  $\mathrm{MHS}_{g,n}$  are determined up to isogeny by Lie algebra cohomology via the following isomorphisms (cf. [20, Cor. 3.7]):

$$\mathrm{Ext}_{\mathrm{MHS}_{g,n}}^{\bullet}(\mathbb{Q}(0), \mathbb{V}) \cong H^{\bullet}(\mathcal{G}_{C,P}, V_{C,P}) \cong H^{\bullet}(\mathfrak{u}_{C,P}, V_{C,P})^{\mathfrak{sp}(H_1(C))}.$$

Although Theorem A.1 was originally proved by a direct argument, it is most natural to regard it as a consequence of this general result and known facts about the cohomology of mapping class and Torelli groups.

The relevance of the algebra  $A_{g,n}^{\bullet}$  defined in the introduction is that there is a natural  $\mathrm{Sp}_g$ -equivariant algebra homomorphism

$$H^{\bullet}(\mathfrak{u}_{C,P}) \rightarrow A_{g,n}^{\bullet}$$

which is an isomorphism in degrees 0 and 1 and injective in degree 2. The construction is described in [17] in a more general context. The algebra  $T_{g,n}^{\bullet}$  is simply the subalgebra of  $A_{g,n}^{\bullet}$  generated by the image of  $H^1(\mathfrak{u})$  — that is, by normal functions.

The coefficients the boundary component  $\delta_h^P$ , ( $h > 0$ ) in the formulas in Section 10 are determined by the image in the second weight graded quotient of  $\mathfrak{u}_{C,P}$  of the Dehn twist corresponding to a loop about  $\Delta_h^P$ . The coefficients of  $\delta_0^P$  can be determined similarly, although one has to work with the appropriate relative weight filtration.

One remaining question is whether more information can be obtained by considering other natural categories of MHS over  $\mathcal{M}_{g,n}$  (possibly with a level structure), such as the category of variations of MHS obtained from the Prym construction.

At present not enough is known about the topology of the Prym construction to understand this problem, although recent progress has been made by Putman [37].

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